

**AN ERROR ANALYSIS OF A PROTOTYPE FREQUENCY SPECTRUM
ANALYZER FOR SIGNALS OF SHORT DURATION**

A Thesis

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CHAPTER I

Various methods may be used to obtain the Fourier analysis of a time function. One common method of analyzing periodic signals is to heterodyne a swept local oscillator with the signal in question.¹ The difference frequency is applied to a narrow band intermediate frequency amplifier and corresponds to the small band being "looked at" by the analyzer at any instant. The IF filter response is then plotted as the amplitude frequency spectrum. Many different commercial spectrum analyzers exist that will determine the frequency components of a periodic signal.

The situation is different, however, when an aperiodic signal of short duration is to be analyzed. This signal may be analyzed by ordinary techniques if first the signal is made periodic. Typical of techniques used to make a transient signal periodic is the one described by Whiteley and Alldredge.

"A transient waveform to be analyzed is made periodic by repetition. The resulting Fourier series components are simply related to the continuous frequency distribution of the original transient waveform. The film on which the transient to be analyzed is recorded is placed in a variable speed revolving lucite drum. A photoelectric system regenerates the signal which is applied to fixed tuned amplifier circuits. Changing the speed of rotation has exactly the same effect as changing the tuning of the analyzer. Hence the analysis is performed by allowing the drum to slow down from maximum speed under frictional forces. By this means a large number of points on the frequency spectrum can be rapidly obtained and presented in the form of a permanent ink record."²

¹Frequency Response Curve Tracing, The Panoramic Analyzer; 2:1-4, Panoramic Radio Products, Inc., Mount Vernon, New York.

²T. B. Whiteley, and L. R. Alldredge, "A Photomechanical Wave Analyzer for Fourier Analysis of Transient Waveforms," Journal of Scientific Instruments, 29:358, June, 1952.

The main disadvantage of analyzing a transient waveform in this manner is the time needed to obtain the frequency amplitude spectrum.

A transient signal frequency spectrum analyzer employing a different technique than that commonly used has been constructed. It obtains samples of the amplitude frequency spectrum of a transient waveform at the termination of the waveform by using the comb filter method. This method uses a separate filter to obtain each spectrum sample. A smooth frequency amplitude spectrum is then constructed from these samples.

The design of the spectrum analyzer was intuitive but comparison of its output and a calculated spectrum demonstrated its accuracy. It is the intended purpose of this study to analyze the methods and circuits used in the spectrum analyzer and to thereby establish the degree of accuracy of results obtained by its use.

CHAPTER II

THE RESPONSE OF A FILTER CIRCUIT

A high Q filter is the circuit that actually measures the frequency spectrum sample. Remaining circuits are used to make the output of the filter available. The impedance of the filter at resonance is calculated. Using the superposition theorem the response of the filter to an arbitrary input is then derived.

2.1 Filter Impedance at Resonance

The filter under consideration may be depicted as the one in Figure 1.

There
$$Z = \frac{Z_L Z_C}{Z_L + Z_C}$$

therefore
$$Z = R + j\omega L \text{ and } Z_C = R_C + \frac{1}{j\omega C}$$

$$Z = \frac{(R_L + j\omega L)(R_C + \frac{1}{j\omega C})}{(R_L + j\omega L) + (R_C + \frac{1}{j\omega C})}$$

$$Z(j\omega) = \frac{R_L R_C + j\omega L R_C - j \frac{R_L + L}{\omega C}}{R_L + R_C + j(\omega L - \frac{1}{\omega C})}$$

This expression for $Z(j\omega)$ is quite complex, but the fact that the circuit has a high Q enables simplification. From the numerator,

$$R_L R_C + \frac{L}{C} + j(\omega L R_C - \frac{R_L}{\omega C})$$

$R_L R_C$ can be disregarded with 1 per cent or less error if

$$R_L R_C \leq .01 \frac{L}{C}$$

The total series resistance remains constant at the resonant frequency of the filter.

$$R_S = R_L + R_C = K$$

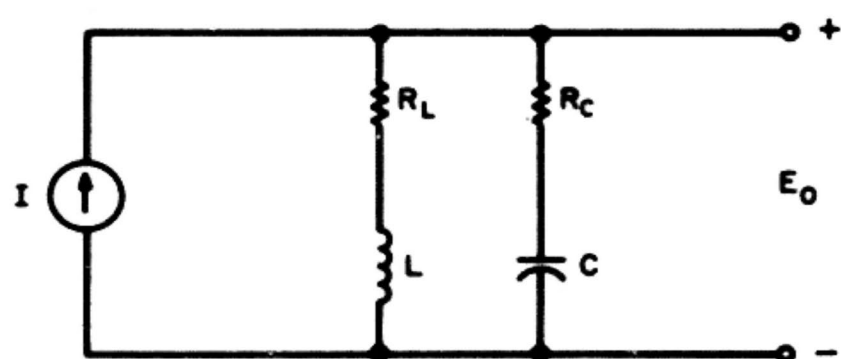


Figure 1. Filter Energized by a Current Source

The value of $R_L R_C$ depends on what part of R_S is due to R_L and what part is due to R_C . The distribution that gives the product its maximum value can be found by differentiation of $R_L R_C$.

$$R_C = R_S - R_L$$

Therefore

$$R_L R_C = R_L R_S - R_L^2$$

$$\frac{d(R_L R_S - R_L^2)}{d R_L} = R_S - 2R_L$$

The maximum value of $R_L R_C$ is found by setting $\frac{d(R_L R_S - R_L^2)}{d R_L}$ equal to zero.

$$R_S - 2R_L = 0$$

$$R_L = R_S/2$$

But

$$R_C = R_S - R_L$$

$$R_C = R_S - R_S/2 = R_S/2$$

For a given R_S , the maximum value of $R_L R_C$ occurs when

$$R_L = R_C = R_S/2$$

Therefore the maximum value of $R_L R_C$ is

$$R_L R_C \text{ max} = R_S^2/4$$

The definition of Q_0 at resonance for parallel resonant circuits is³

$$Q_0 = \frac{\omega_0 L}{R_S}$$

³F. E. Terman, Radio Engineers Handbook (New York: McGraw-Hill Book Company, Inc., 1943), p. 141.

Therefore $R_s = \frac{\omega_0 L}{Q_0}$

And $R_L R_{Cmax} = \frac{\omega_0^2 L^2}{4Q_0^2}$

So that when $R_L R_{Cmax} = \frac{\omega_0^2 L^2}{4Q_0^2} \leq 0.01 \frac{L}{C}$

$$0.01 Q_0^2 \geq \frac{\omega_0^2 L^2 C}{4L} = \frac{\omega_0^2 L C}{4}$$

Taking the square root of both sides

$$0.1 Q_0 \geq \omega_0 \sqrt{LC}/2$$

The resonant frequency of a parallel resonant circuit with moderate or high Q_0 is⁴

$$\omega_0 = 1/\sqrt{LC} \text{ when } Q_0 \geq 10$$

therefore $0.1 Q_0 \geq 1/2$

$$Q_0 \geq 10/2 = 5$$

This means that if 1 percent accuracy is needed and it is advantageous to ignore the $R_L R_C$ term the Q_0 has to have a value of at least five.

The $j(\omega_0 L R_C - R_L/\omega_0 C)$ term can be disregarded with 1 percent accuracy if $|\omega_0 L R_C - R_L/\omega_0 C| \leq 0.1414 L/C$. Neither $\omega_0 L R_C$ nor $R_L/\omega_0 C$ can be negative numbers. Therefore the differences of the two must be less than or equal to one of the terms. Either

$$\omega_0 L R_C \geq |\omega_0 L R_C - R_L/\omega_0 C| \text{ or}$$

$$R_L/\omega_0 C \geq |\omega_0 L R_C - R_L/\omega_0 C|$$

If conditions can be found such that both $\omega_0 L R_C$ and $R_L/\omega_0 C$ can be neglected without losing more than 1 percent accuracy then $\omega_0 L R_C - R_L/\omega_0 C$ can also be neglected losing 1 percent or less accuracy.

$\omega_0 L R_C$ must $\leq 0.1414 L/C$ so that the $j \omega_0 L R_C$ term can be neglected.

⁴Ibid., p. 144.

$$\omega_0 \leq 0.1414 L / LCR_C = 0.1414 / CR_C$$

Here the upper bound for ω_0 is set so for the worst possible case,

$$R_C = R_S \text{ or } R_L = 0.$$

$$\text{Then } \omega_0 \leq 0.1414 / CR_S \text{ and since } R_S = \omega_0 L / Q_0$$

$$\omega_0 \leq 0.1414 Q_0 / C \omega_0 L$$

$$\text{or } \omega_0^2 \leq 0.1414 Q_0 / CL \text{ but since } \omega_0^2 = 1 / LC$$

$$Q_0 \geq 1 / 0.1414 = 7.1$$

$$\text{If } R_L / \omega_0 C \leq 0.1414 L / C$$

Then the $jR_L / \omega_0 C$ term can be neglected.

$$\omega_0 \geq 0.1414 R_L C / LC = 0.1414 R_L / L$$

Here the lower bound for ω_0 is set for the worst possible case,

$$R_L = R_S \text{ or } R_C = 0$$

$$\text{Then } \omega_0 \geq 0.1414 R_S / L \text{ and since } R_S = \omega_0 L / Q_0$$

$$\omega_0 \geq 0.1414 \omega_0 L / Q_0 L$$

$$1 \geq 0.1414 / Q_0$$

$$\text{or } Q_0 \geq 7.1$$

Thus both the terms can be neglected if $Q_0 \geq 7.1$ and therefore by previous reasoning $|\omega_0 L R_C - R_L / \omega_0 C| \leq 0.1414 L / C$ if $Q_0 \geq 7.1$. With 1 percent accuracy if Q_0 is greater than or equal to ten at resonance

$$Z(j\omega) = \frac{L/C}{R_L + R_C + j\omega_0 L + 1/j\omega_0 C}$$

$$Z(j\omega) = \frac{j\omega_0 CL/C}{(j\omega_0)^2 LC + j\omega_0 C(R_L + R_C) + 1}$$

$$\text{Letting } j\omega_0 = S$$

$$Z(S) = \frac{SL}{S^2 LC + SCR_S + 1}$$

$$Z(S) = \frac{S}{C} \frac{1}{S^2 + SR_S/L + 1/LC} = \frac{S}{C} \frac{1}{S^2 + S\omega_0/Q_0 + \omega_0^2}$$

(1)

Hereafter it shall be assumed that Q_0 is greater than or equal to ten and the accuracy will be within 1 percent.

2.2 Response to a Step Input

Computing the response of the filter to a unit step function of current:

$$I(S) = 1/S$$

$$E_o(S) = I(S) Z(S)$$

$$E_o(S) = \frac{1}{S} \cdot \frac{S}{C} \cdot \frac{1}{S^2 + SR_S/L + 1/LC}$$

$$E_o(S) = \frac{1}{C} \frac{1}{S^2 + SR_S/L + 1/LC}$$

Using Laplace transform tables⁵ to find $E_o(t)$ results in

$$E_o(t) = \frac{1}{C} \cdot \frac{1}{\sqrt{1/LC - R_S^2/4L^2}} e^{-(R_S/2L)t} \sin \sqrt{1/LC - R_S^2/4L^2} t$$

Letting $R_S/2L = \alpha$ and $\sqrt{1/LC - R_S^2/4L^2} = \omega$

$$\alpha = R_S/2L = \omega_0/2Q_0$$

$$\omega = \sqrt{1/LC - R_S^2/4L^2} = \sqrt{\omega_0^2 - \alpha^2}$$

$$\omega = \sqrt{\omega_0^2 - \omega_0^2/4Q_0^2} = \omega_0 \sqrt{1 - 1/4Q_0^2}$$

$$\sqrt{1 - 1/4Q_0^2} \approx 1 \text{ since } Q_0 \geq 10$$

Therefore

$$\omega \approx \omega_0$$

$$E_o(t) \approx \frac{1}{C\omega_0} e^{-\frac{\omega_0}{2Q_0} t} \sin \omega_0 t \quad (2)$$

The output of the filter with a unit step function of current as the input will be a damped sine wave.

⁵W. F. Thomson, Laplace Transformation (New Jersey: Prentice-Hall, Inc., 1960), p. 243.

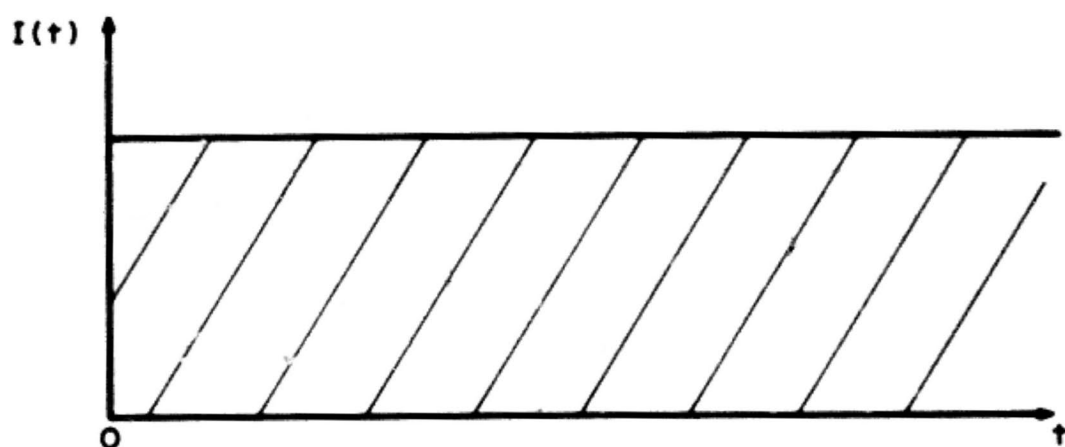


Figure 2. Unit Step Function

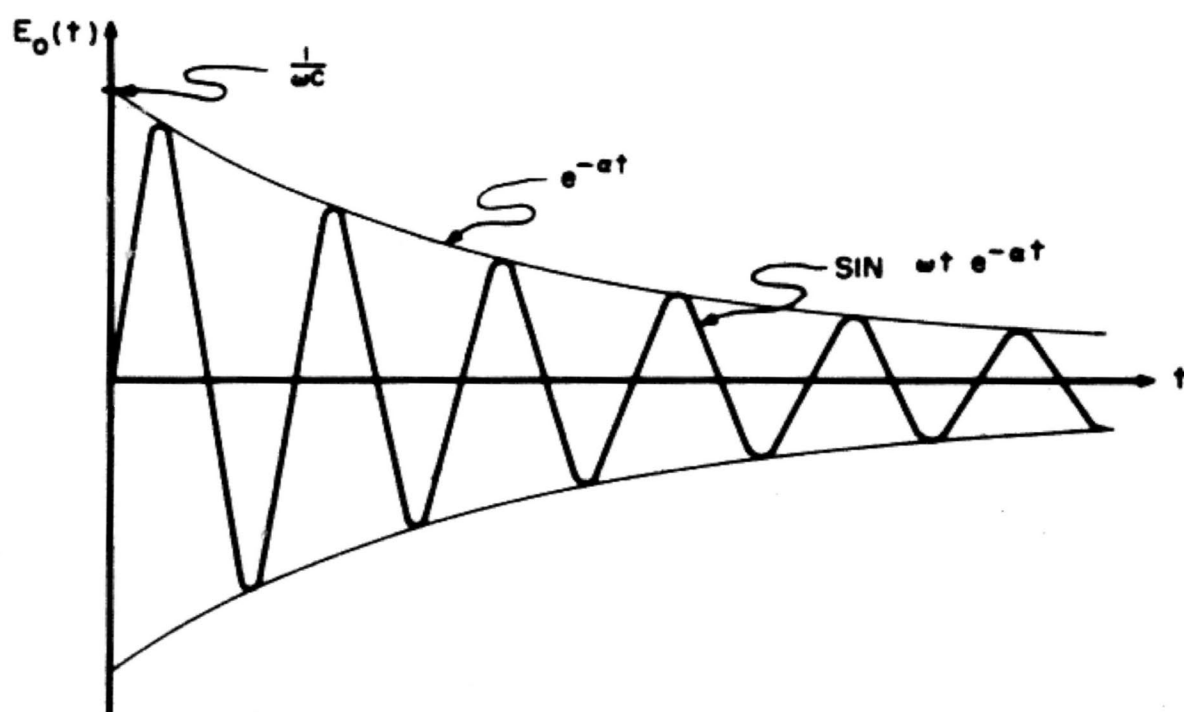


Figure 3. Filter Output Due to Step Input

2.3 Response to an Arbitrary Input

To find the response of the filter to an arbitrary input, the input must first be defined. As conditions of $I(t)$, let

$$I(t) = \begin{cases} 0, & t < 0 \\ 0, & t \geq T \end{cases}$$

and the integral $\int_0^T I(t) dt$ be finite

A function satisfying these conditions would be one as in Figure 4. This function can be approximated by a series of step functions.

$$I(t) \approx I(0+) U(t) + [I(\Delta t) - I(0+)] U(t - \Delta t) + [I(2\Delta t) - I(\Delta t)] U(t - 2\Delta t) + \dots$$

$$\dots + [I(n\Delta t) - I((n-1)\Delta t)] U(t - n\Delta t) + \dots$$

$$I(t) \approx I(0+) U(t) + \sum_{n=1}^{n=N} [I(n\Delta t) - I((n-1)\Delta t)] U(t - n\Delta t)$$

If $I(t)$ is the input to the filter, then its output will be

$$E_o(t) \approx \frac{I(0+)}{C\omega} e^{-at} \sin \omega t + \frac{1}{C\omega} \sum_{n=1}^{n=N} [I(n\Delta t) - I((n-1)\Delta t)] e^{-a(t-n\Delta t)} \sin \omega(t-n\Delta t)$$

This approximation becomes better as N approaches infinity which also makes Δt approach zero. When the limit exists (one of the conditions on $I(t)$) then the equation becomes

$$E_o(t) = \frac{I(0+)}{C\omega} e^{-at} \sin \omega t + \frac{1}{C\omega} \int_0^T \frac{dI(\tau)}{d\tau} \sin \omega(t-\tau) e^{-a(t-\tau)} d\tau$$

This expression can be integrated using the method of integration by parts.

Let

$$du = dI(\tau)$$

$$u = I(\tau)$$

$$v = \sin \omega(t-\tau) e^{-a(t-\tau)}$$

$$dv = [-\omega \cos \omega(t-\tau) e^{-a(t-\tau)} + a \sin \omega(t-\tau) e^{-a(t-\tau)}] d\tau$$

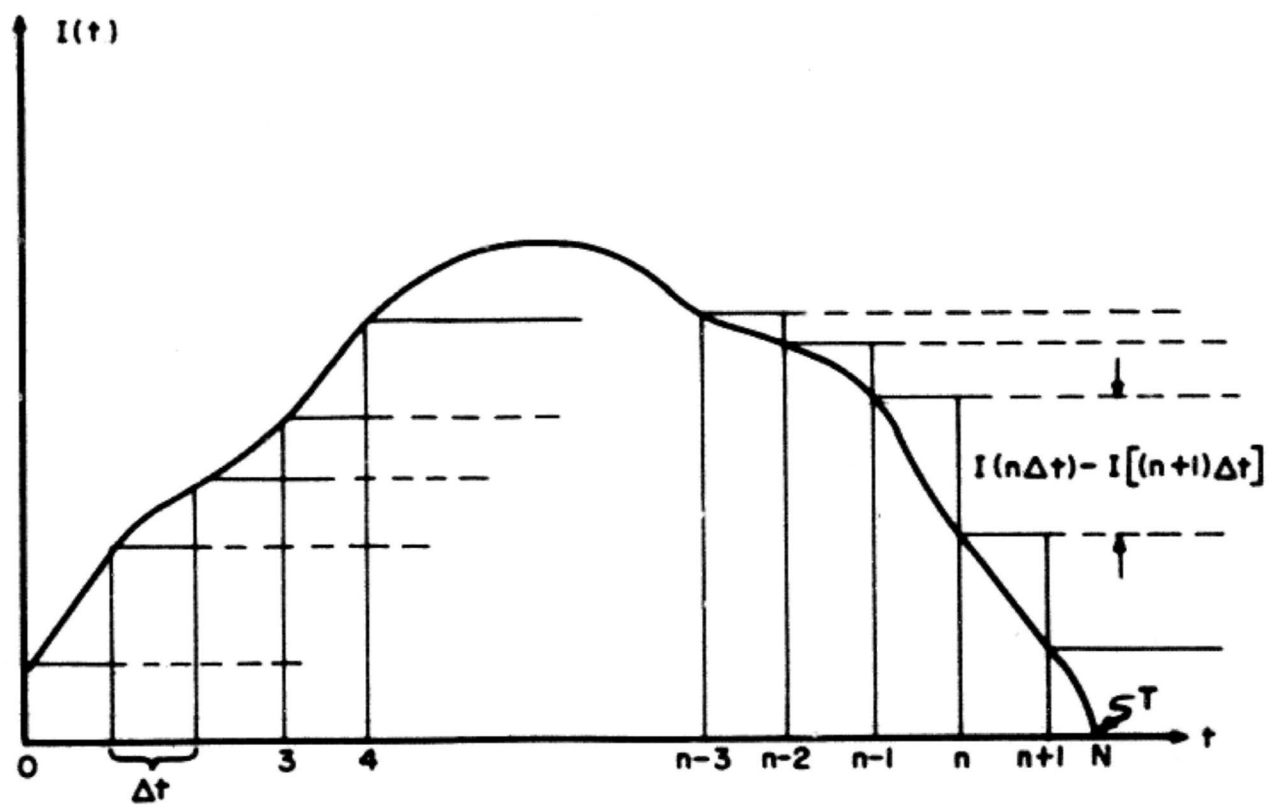


Figure 4. An Arbitrary Input

Then
$$E_o(t) = \frac{I(o+)}{C\omega} e^{-at} \sin \omega t + \frac{1}{C\omega} [I(\tau) \sin \omega(t-\tau) e^{-a(t-\tau)}]_0^T + \frac{\omega}{C\omega} \int_0^T I(\tau) \cos \omega(t-\tau) e^{-a(t-\tau)} d\tau - \frac{a}{C\omega} \int_0^T I(\tau) \sin \omega(t-\tau) e^{-a(t-\tau)} d\tau$$

Simplifying the first part of this equation:

$$\frac{I(o+)}{C\omega} e^{-at} \sin \omega t + \frac{I(T)}{C\omega} e^{-a(t-T)} \sin \omega(t-T) - \frac{I(o+)}{C\omega} e^{-at} \sin \omega t$$

$I(T)$ equals zero by definition and so this portion of $E_o(t)$ is equal to zero. $E_o(t)$ is therefore equal to the remaining portion of the equation.

$$E_o(t) = \frac{1}{C} \int_0^T I(\tau) \cos \omega(t-\tau) e^{-a(t-\tau)} d\tau - \frac{a}{C\omega} \int_0^T I(\tau) \sin \omega(t-\tau) e^{-a(t-\tau)} d\tau$$

Expanding the $\cos \omega(t-\tau)$ and $\sin \omega(t-\tau)$ terms

$$E_o(t) = \frac{1}{C} \int_0^T I(\tau) [\cos \omega t \cos \omega \tau + \sin \omega t \sin \omega \tau] e^{-at} e^{a\tau} d\tau - \frac{a}{\omega} \int_0^T I(\tau) [\sin \omega t \cos \omega \tau - \cos \omega t \sin \omega \tau] e^{-at} e^{a\tau} d\tau$$

Separating the variables

$$E_o(t) = \frac{e^{-at}}{C} \left\{ \cos \omega t \int_0^T I(\tau) \cos \omega \tau e^{a\tau} d\tau + \sin \omega t \int_0^T I(\tau) \sin \omega \tau e^{a\tau} d\tau \right\} - \frac{ae^{-at}}{C\omega} \left\{ \sin \omega t \int_0^T I(\tau) \cos \omega \tau e^{a\tau} d\tau - \cos \omega t \int_0^T I(\tau) \sin \omega \tau e^{a\tau} d\tau \right\}$$

Rearranging terms

$$E_o(t) = \frac{e^{-at}}{C} \left\{ \left[\int_0^T I(\tau) \cos \omega \tau e^{a\tau} d\tau + \frac{a}{\omega} \int_0^T I(\tau) \sin \omega \tau e^{a\tau} d\tau \right] \cos \omega t + \left[\int_0^T I(\tau) \sin \omega \tau e^{a\tau} d\tau - \frac{a}{\omega} \int_0^T I(\tau) \cos \omega \tau e^{a\tau} d\tau \right] \sin \omega t \right\} \quad (3)$$

Let
$$\int_0^T I(\tau) \cos \omega \tau e^{a\tau} d\tau + \frac{a}{\omega} \int_0^T I(\tau) \sin \omega \tau e^{a\tau} d\tau = a(\omega)$$

and
$$\int_0^T I(\tau) \sin \omega \tau e^{a\tau} d\tau - \frac{a}{\omega} \int_0^T I(\tau) \cos \omega \tau e^{a\tau} d\tau = b(\omega)$$

Then the expression for the output voltage becomes

$$E_o(t) = \frac{e^{-\alpha t}}{C} [a(\omega) \cos \omega t + b(\omega) \sin \omega t]$$

For the ideal filter, alpha equals zero as there is no damping so that $E_o(t) = \frac{1}{C} [a(\omega) \cos \omega t + b(\omega) \sin \omega t]$

where $a(\omega) = \int_0^T I(\tau) \cos \omega \tau d\tau$ and $b(\omega) = \int_0^T I(\tau) \sin \omega \tau d\tau$

This compares to the Fourier Integral formula⁶ which is

$$G(t) = \int_0^{\infty} a(\omega) \cos \omega t d\omega + \int_0^{\infty} b(\omega) \sin \omega t d\omega$$

$$G(t) = \int_0^{\infty} [a(\omega) \cos \omega t + b(\omega) \sin \omega t] d\omega$$

For the ideal filter $E_o(t)$ is the integrand of the Fourier integral.

The Fourier integral can also be expressed in the complex form as

$$G(t) = \int_{-\infty}^{+\infty} e^{j\omega t} F(f) df$$

where $F(f) = \int_{-\infty}^{+\infty} e^{-j\omega t} G(t) dt$ = the Fourier integral

As $E_o(t)$ is the integrand of the Fourier integral

$$E_o(t) = e^{j\omega t} F(f)$$

or

$$|E_o(t)| = F(f)$$

Where $F(f)$ is the value of the Fourier integral at frequency f .

Thus the absolute value of the output voltage is a sample of the Fourier integral.

⁶S. Goldman, Frequency Analysis, Modulation and Noise (New York: McGraw-Hill Book Company, Inc., 1948) p. 54.

⁷Ibid., pp. 58-9.

2.4 Number of Filters Required

An infinite number of samples taken at frequencies ranging from zero radians per second to infinity could be used to find the complete frequency spectrum of the input. The frequency spectrum constructed from the outputs of actual filters will have some error due to circuit losses.

It is impractical to use an infinite number of filters to find the Fourier integral. Appendix A shows that the frequency spectrum of a transient waveform contained in an interval of one millisecond can be found from samples spaced one kilocycle apart. This however necessitates the optimum use of an infinite number of these samples. Even if the waveform is band limited to contain frequencies only to perhaps one hundred kilocycles, the equation for the frequency spectrum involves operating on each of one hundred samples and then summing them. This is also quite a task.

The condition that the frequency spectrum be accurately produced by connecting the samples with smooth lines requires samples spaced closer together. The type of waveform dictates the spacing of the samples. Sampling rate in the time domain is usually at least three times the minimum sampling rate obtained by using Shannon's sampling theorem.⁶ This practice can also be followed in the frequency domain. Then a first approximation, if the type of waveform contained in the interval T is unknown, might be to take samples every $3/T$ cycles. The spacing between samples then is one third of the maximum spacing $1/T$ cycles. Closer spacing of the samples would be indicated if the spectrum curve obtained by joining successive samples with straight lines is very discontinuous.

⁶B. M. Oliver, J. R. Pierce and C. E. Shannon, "The Philosophy of PCM", Proceedings of the IRE, 36:1330, November, 1948

CHAPTER III

DETERMINATION OF RELATIONSHIP BETWEEN CIRCUIT ELEMENTS AND ERROR

Since lossless, infinite Q_0 , circuits are physically unrealizable, the response of a physical filter will contain an error. The losses in the filter will set the value of α , the damping factor. $\alpha = \frac{\omega_0}{2Q_0}$

So for Q_0 equal to infinity, α equals zero.

The amplitude error due to finite circuit Q_0 and a given input signal can be evaluated. The relative error will be the ratio of the difference between the response of an infinite Q_0 circuit and a finite Q_0 circuit to the response of an infinite Q_0 circuit.

3.1 Error Due to a Step Input

Consider a step input signal. Equation (2) gives the response of the filter to a unit step. If Q_0 equals infinity, α equals zero, then

$$E_O(t)_I = \frac{1}{\omega_0 C} \sin \omega_0 t$$

with a finite Q_0 ,
$$E_O(t)_F = \frac{1}{\omega C} e^{-\alpha t} \sin \omega t$$

For an amplitude frequency spectrum only the peak amplitude of the response is important.

$$E_{OI} = \frac{1}{\omega_0 C}$$

$$E_{OF} = \frac{1}{\omega C} e^{-\alpha t}$$

$$E_{OF} = \frac{1}{C\omega_0 \sqrt{1 - 1/4 Q_0^2}} e^{-\frac{\omega_0}{2Q_0} t}$$

The relative error is

$$\delta = \frac{\frac{1}{\omega_0 C} - \frac{1}{\omega_0 C \sqrt{1 - 1/4 Q_0^2}} e^{-\frac{\omega_0}{2Q_0} t}}{\frac{1}{\omega_0 C}}$$

$$\delta \approx 1 - \frac{e^{-\frac{\omega_0}{2Q_0}t}}{\sqrt{1 - 1/4Q_0^2}}$$

A previous assumption is

$$Q_0 \geq 10$$

$$\text{Therefore } \sqrt{1 - 1/4Q_0^2} \leq \sqrt{1 - 1/400} = \sqrt{0.9975}$$

$$\sqrt{1 - 1/4Q_0^2} \approx 1$$

$$\delta \approx 1 - e^{-\frac{\omega_0}{2Q_0}t}$$

$$e^{-\frac{\omega_0}{2Q_0}t} \approx 1 - \delta$$

$$-\frac{\omega_0}{2Q_0}t = \ln(1 - \delta)$$

Taylor's expansion of e^{-x} is

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots$$

When the error is small only the first two terms of the series are significant.

$$e^{-x} \approx 1 - x$$

therefore

$$-\frac{\omega_0}{2Q_0}t \approx \ln(e^{-\delta})$$

$$-\frac{\omega_0}{2Q_0}t \approx -\delta$$

$$\delta \approx \frac{\omega_0}{2Q_0}t \quad (4)$$

The error can be expressed in terms of resonant frequency, the resonant Q of the circuit and the time at which the amplitude is measured.

The resonant frequency, the time at which the filter is read and the allowable error are constants, only Q_0 is variable. As an example, let ω_0 equal ten kilocycles, t equal one millisecond and the allowable error be 1 percent. Then

$$.01 \cong \frac{10^4}{2Q_0} \times 2\pi \times 10^{-3}$$

$$.01 \cong \frac{31.4}{Q_0}$$

$$Q_0 \cong 3,140$$

Intuitively, it seems that the largest possible error will occur when the input signal is a step function. This is because all of the information contained in the signal occurs at t equal to zero. Therefore the filter response will be damped the maximum amount. A signal containing information throughout its duration will have the response due to the initial portion damped more than the response due to later portions. This theory can be checked by finding the error when the input is a gated sine wave which contains information throughout its duration

3.2 Error Due to a Gated Sine Wave Input

Consider the error when the input is a gated sine wave consisting of N cycles.

The filter response to any input is given by the integral formula

$$(3) \quad E_0(t) = \frac{e^{-at}}{C} \left\{ \left[\int_0^T I(\tau) \cos \omega \tau e^{a\tau} d\tau + \frac{a}{\omega} \int_0^T I(\tau) \sin \omega \tau e^{a\tau} d\tau \right] \cos \omega t \right. \\ \left. + \left[\int_0^T I(\tau) \sin \omega \tau e^{a\tau} d\tau - \frac{a}{\omega} \int_0^T I(\tau) \cos \omega \tau e^{a\tau} d\tau \right] \sin \omega t \right\}$$

$$E_0(t) = \frac{e^{-at}}{C} \left\{ \left(I_1 + \frac{a}{\omega} I_2 \right) \cos \omega t + \left(I_2 - \frac{a}{\omega} I_1 \right) \sin \omega t \right\}$$

where

$$I_1 = \int_0^T I(\tau) \cos \omega \tau e^{a\tau} d\tau \text{ and}$$

$$I_2 = \int_0^T I(\tau) \sin \omega \tau e^{a\tau} d\tau$$

Performing the indicated integration when $I(t)$ equals $\sin \omega t$, (see Appendix B)

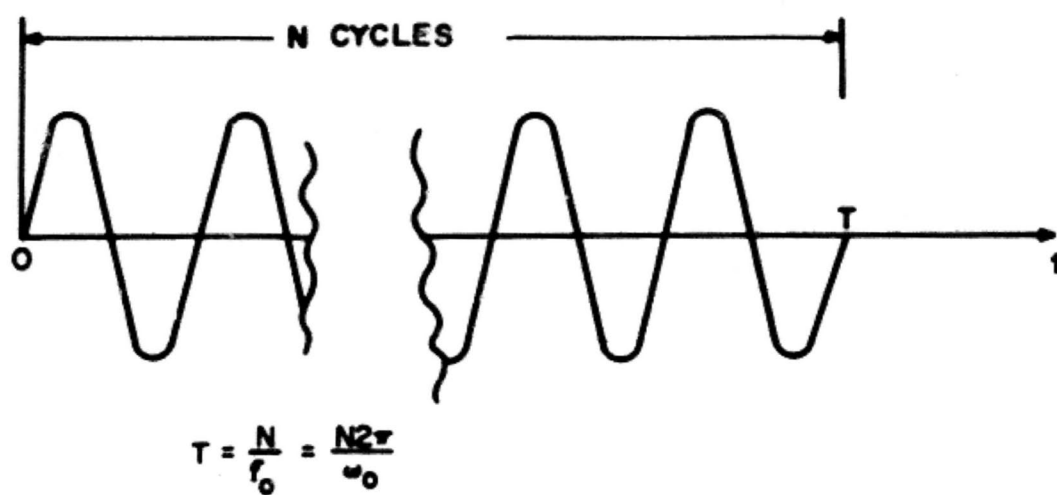


Figure 5. Gated Sine Wave Input

$$E_o(t) = \frac{e^{-at}(e^{aT} - 1)}{2aC} \sin \omega t$$

$$E_{oF} = \frac{e^{-at}}{2aC} (e^{aT} - 1)$$

To get response of lossless filter let $a = 0$

$$\text{Then } E_{oI} = \lim_{a \rightarrow 0} \frac{e^{-a(t-T)} - e^{-at}}{2aC} = \lim_{a \rightarrow 0} \frac{\frac{d[e^{-a(t-T)} - e^{-at}]}{da}}{\frac{d(2aC)}{da}}$$

by applying l'Hospital's rule since this is an indeterminate form

$$E_{oI} = \lim_{a \rightarrow 0} \frac{e^{-a(t-T)}(-t+T) + te^{-at}}{2C} = \frac{-t+T+t}{2C} = \frac{T}{C}$$

$$\text{Now } \delta = \frac{E_{oI} - E_{oF}}{E_{oI}} = \frac{\frac{T}{2C} - \frac{e^{-at}(e^{aT} - 1)}{2aC}}{T/2C}$$

$$\delta = 1 - \frac{e^{-at}(e^{aT} - 1)}{Ta}$$

$$\text{But } T = \frac{N 2\pi}{\omega_o} \text{ and } a = \frac{\omega_o}{2Q_o}$$

$$\delta = 1 - \frac{e^{-\frac{\omega_o}{2Q_o} t} (e^{\frac{N\pi}{\omega_o}} - 1)}{\frac{N\pi}{Q_o}}$$

A reasonable place to evaluate the relative error would be at the end of the signal or when t equals T . From that time until the time the filter is read, the total response is damped by a term $e^{-a(t-T)}$. Before this time, responses to different portions of the signal were damped different amounts.

$$\text{At } t = T \quad \delta = 1 - \frac{Q_o}{N\pi} e^{-\frac{N\pi}{Q_o}} (e^{\frac{N\pi}{Q_o}} - 1)$$

$$\delta = 1 - \frac{Q_0}{N\pi} (1 - e^{-\frac{N\pi}{Q_0}})$$

$$\begin{aligned} (1 - e^{-\frac{N\pi}{Q_0}}) &= 1 - [1 - \frac{N\pi}{Q_0} + \left(\frac{N\pi}{Q_0}\right)^2 \frac{1}{2!} - \left(\frac{N\pi}{Q_0}\right)^3 \frac{1}{3!} + \dots] \\ &= [\frac{N\pi}{Q_0} - \left(\frac{N\pi}{Q_0}\right)^2 \frac{1}{2!} + \left(\frac{N\pi}{Q_0}\right)^3 \frac{1}{3!} - \dots] \end{aligned}$$

$$\text{So that } \delta = 1 - \frac{Q_0}{N\pi} [\frac{N\pi}{Q_0} - \left(\frac{N\pi}{Q_0}\right)^2 \frac{1}{2!} + \left(\frac{N\pi}{Q_0}\right)^3 \frac{1}{3!} - \dots]$$

$$\delta = 1 - [1 - \frac{N\pi}{Q_0} \frac{1}{2!} + \left(\frac{N\pi}{Q_0}\right)^2 \frac{1}{3!} - \left(\frac{N\pi}{Q_0}\right)^3 \frac{1}{4!} + \dots]$$

$$\delta = [\frac{N\pi}{Q_0} \frac{1}{2!} - \left(\frac{N\pi}{Q_0}\right)^2 \frac{1}{3!} + \left(\frac{N\pi}{Q_0}\right)^3 \frac{1}{4!} - \dots]$$

$$\delta = \sum_{n=1}^{n=\infty} \left(\frac{N\pi}{Q_0}\right)^n \frac{1}{(n+1)!}$$

For a small relative error only the first term of the series is significant.

$$\delta \approx \frac{N\pi}{Q_0} \frac{1}{2}$$

If the input consisted of ten cycles of the sine wave and an accuracy of 1 percent is desired, then the conditions for Q_0 are set.

$$.01 \approx \frac{10\pi}{Q_0} \frac{1}{2}$$

$$Q_0 \approx \frac{5\pi}{.01} = 500\pi$$

$$Q_0 \approx 1,570$$

The times at which the errors for the step and sine wave inputs to the ten kilocycle channel were checked were equivalent. The time was one millisecond or equal to the time taken by ten cycles of a ten kilocycle sine wave. Since the times were equal the limits on Q_0 in each case could be compared. For an error of 1 percent with a step

input the Q_0 had to be at least 3,140. An allowable error of 1 percent with a sine wave input required a Q_0 of only 1,570, half that required by a step input. These results would agree with the surmise that the maximum error would occur with a step input. Another way this could be put is that the error for any arbitrary input is always less than or equal to the error caused by a step input.

Something could be said about the phase error contained in the filter response. An error could be due to the fact that the frequency of the filter output was ω instead of ω_0 . If this were a phase error it would be negligible at the high Q 's necessary to keep the amplitude error small. However, the filters must be tuned so that the frequency physically present, ω , is equal to the wanted frequency. If a filter was to be tuned to ten kilocycles ω would equal ten kilocycles and since

$$\omega = \omega_0 \sqrt{1 - 1/4Q_0^2}$$

ω_0 would be something less than ten kilocycles. Therefore a perfectly tuned filter would have no phase error even though it had losses.

CHAPTER IV

INSTRUMENTATION

The prototype frequency spectrum analyzer was designed to perform Fourier analysis on a transient signal that was contained in one millisecond of time. Supporting equipment delayed the transient signal so that a gate pulse could be formed to start 300 microseconds before the peak of the delayed transient signal. The signal was also normalized so that the peak to peak amplitude of the signal would vary between ten and 31.6 volts.

A spectrum analyzer control circuit received the gate pulse and from it formed three other pulses. One of the pulses was the filter unclamping pulse starting at time zero (where time zero is at the start of the gate pulse) and existing for 1200 microseconds. Another was the switching pulse starting at time zero and existing for one millisecond and the third was the memory capacitor unclamping pulse starting at time equal to one millisecond and existing for 900 microseconds. These were the outputs of the control circuit.

The spectrum analyzer had as inputs the normalized transient signal and the three pulses from the control circuit.

The spectrum analyzer was designed to produce sixteen samples of the amplitude frequency spectrum. The samples were spaced one kilocycle apart from four kilocycles through fifteen kilocycles. Four additional samples were taken at twenty-five, fifty, seventy-five and one hundred kilocycles.

The frequencies and spacing of the samples that were taken were chosen because of the types of waveforms to be investigated. Previous work on the transient waveforms showed that the spacing between samples was adequate to form a smooth spectrum from the samples.

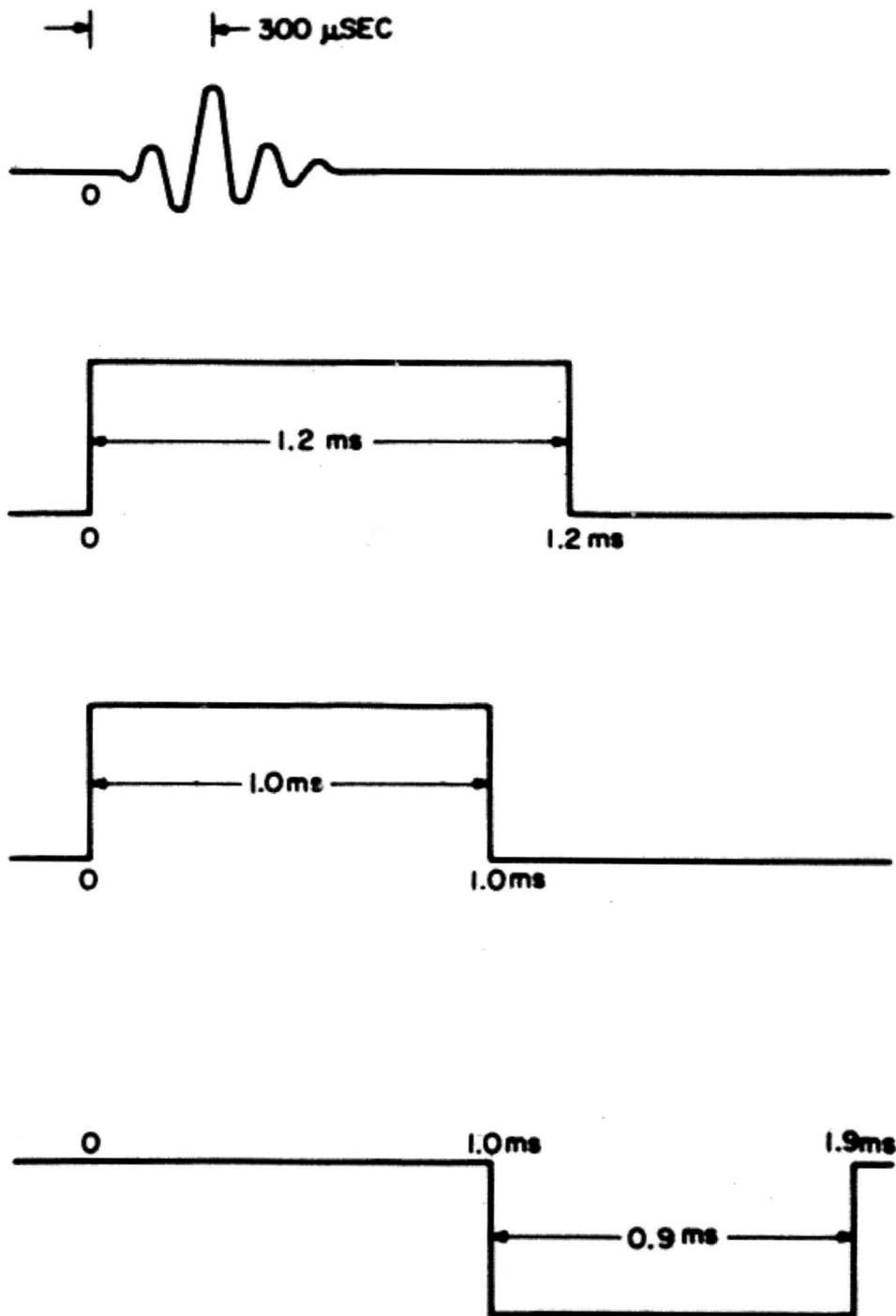


Figure 6. Inputs to Spectrum Analyzer

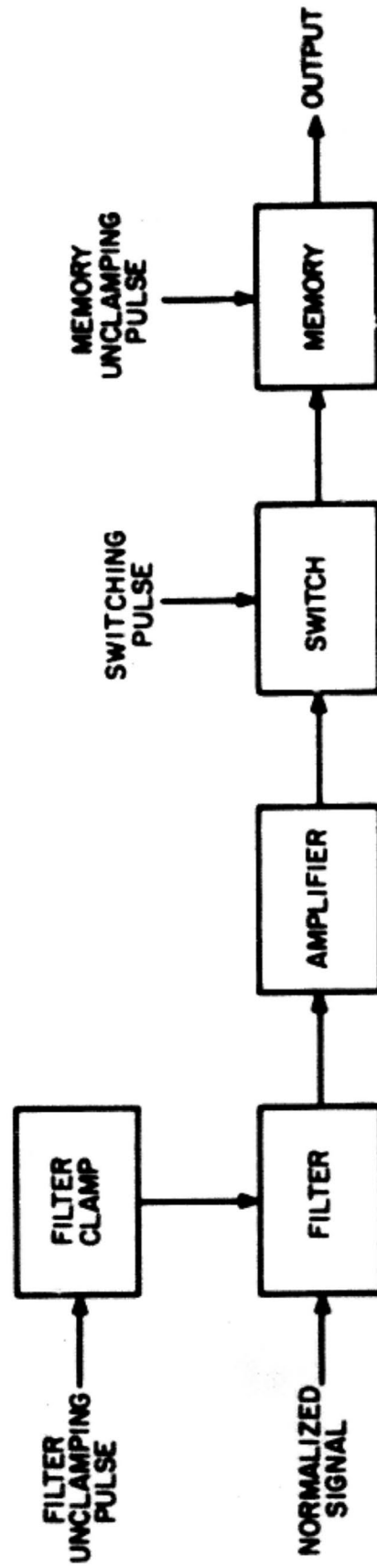


Figure 7. Spectrum Analyzer Channel Block Diagram

Most of the information was contained in frequencies below fifteen kilocycles. The higher frequencies were added for observation.

The previous chapter demonstrated that the cause of error in the response of a physical filter is the damping factor, $e^{-\alpha t}$. Since $\alpha = \frac{\omega_0}{2Q_0}$ a method of keeping the error small suggests itself. If Q_0 varies directly with ω_0 , that is

$$Q_0 = k\omega_0$$

$$\text{then } \alpha = \frac{\omega_0}{2k\omega_0} = \frac{1}{2k}$$

This is a constant for all of the filters regardless of center frequency. If all of the filters have the same damping factor the relative magnitude will remain unchanged with time. Therefore, if the error in the reconstructed waveform were due only to the damping term, it could be eliminated by making Q_0 a function of ω_0 . This would necessitate having a close control on the Q_0 of the filter.

4.1 Basic Circuit

The normalized signal was applied directly to the filter circuit. The filter was the load on the input stage and positive feedback was employed to control its Q_0 .

For a quick analysis the filter can be drawn as in Figure 8. An equivalent circuit looking back into the pentode is to the left of AA' in Figure 9. This can be simplified as in Figure 10 and then redrawn as a current source (Figure 11).

Including the cathode follower's equivalent circuit the filter circuit then appears as in Figure 12.

$$R_1 = \frac{1}{r_p + (\mu + 1)R_k} + \frac{1}{R_g} + \frac{\omega_0^2 L^2}{R_S}$$

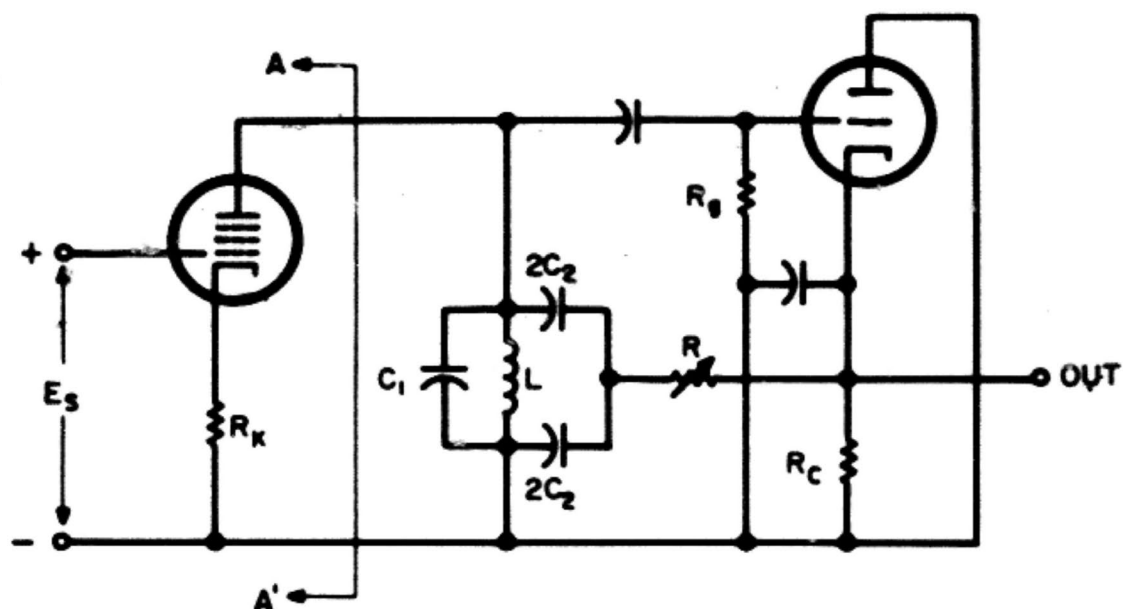


Figure 8. High Q Filter Circuit

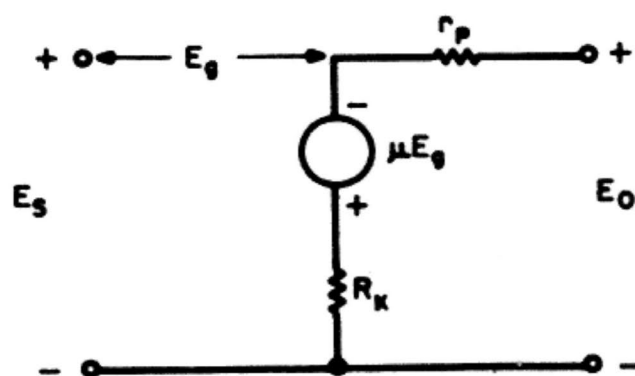


Figure 9. Equivalent Circuit of Pentode Circuit

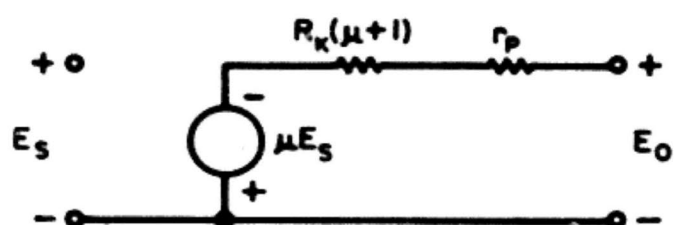


Figure 10. Modified Equivalent Circuit of Pentode Circuit

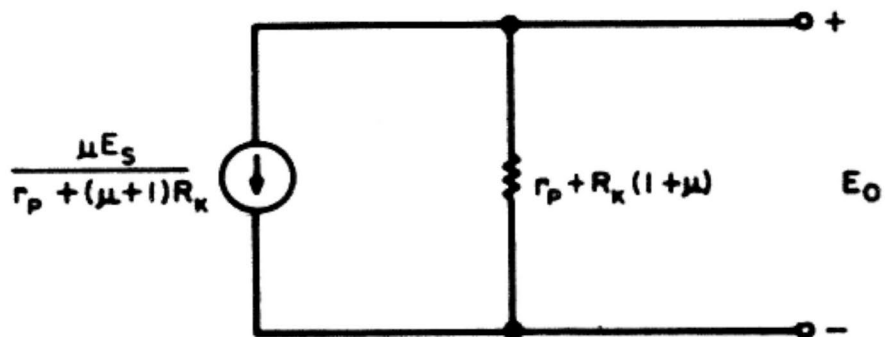


Figure 11. Current Source Equivalent of Pentode Circuit

The impedance looking to the right of BB' can be found by Blackman's formula.⁹

$$\frac{Z_A}{Z_P} = \frac{1 - F_{sh}}{1 - F_{op}}$$

Z_A is the impedance between BB' with feedback.

Z_P is the impedance between BB' without feedback.

F_{sh} is the feedback to the vacuum tube with the BB' terminals shorted.

F_{op} is the feedback to the vacuum tube with the BB' terminals open.

Let μ equal zero to find the impedance without feedback.

Figure 13 shows the circuit without feedback.

Then

$$y_P = \frac{1}{R_1} + \frac{1}{S L} + \frac{1}{\frac{1}{S^2 C} + \frac{1}{S^2 C + \frac{1}{R}}}$$

$$y_P = \frac{S^3 4 C^2 R L R_1 + S^2 2 C L (R_1 + 2 R) + S^4 C R R_1 + S L + R_1}{S^2 4 C L R R_1 + S L R_1}$$

At resonance $j \omega_0 L = j / \omega_0 C$ or $S L = -1 / S C$

Therefore

$$\frac{-S^2 4 C^2 R R_1 - S^2 C (R_1 + 2 R) + S^2 4 C^2 R R_1 - 1 + S C R_1}{S C}$$

$$\frac{-S^4 C R R_1 - R_1}{S C}$$

$$Z_P = R_1 \left[\frac{S^4 C R + 1}{S C R_1 + S^4 C R + 1} \right]$$

$$Z_P = R_1 \left[\frac{4 R / R_1 + 1 / S C R_1}{R / R_1 + 4 R / R_1 + 1 / S C R_1} \right]$$

⁹R. B. Blackman, "Effect of Feedback on Impedance," Bell Systems Technical Journal, 22:272, October, 1943.

$$Z_p = R_1 \left[\frac{4R/R_1 - SL/R_1}{R_1/R_1 + 4R/R_1 - SL/R_1} \right]$$

SL/R_1 is the $1/Q_0$ of the parallel circuit without feedback. Since circuit Q_0 is necessarily high, this term can be neglected.

$$Z_p \approx \left[R_1 \frac{4R}{R_1 + 4R} \right]$$

F_{sh} is feedback to the tube when BB' is shorted. This means E_1 equals zero and then there is no feedback.

$$F_{sh} = 0$$

F_{op} can be found by putting one volt into the tube and determining the voltage that is fed back.

The impedance looking to the right of cc' in Figure 14 is

$$y = \frac{1}{S^2C} + \frac{1}{\frac{1}{S^2C} + \frac{1}{\frac{1}{R_1} + \frac{1}{SL}}}$$

$$y = \frac{\frac{1}{SL} + \frac{1}{R_1} + \frac{1}{S^2C}}{1 + \frac{1}{\frac{2S^2CL}{SL} + \frac{1}{2SCR_1}}}$$

At resonance

$$SC^2 = -2/SL$$

and

$$\frac{1}{2S^2CL} = -\frac{SC}{2SC} = -\frac{1}{2}, \quad \frac{1}{2SCR_1} = -\frac{SL}{2R_1} = -\frac{1}{2Q_0}$$

Therefore

$$y = \frac{2/SL - 2/SL + 2/R_1}{1 - 1/2 - 1/2Q_0}$$

$$Z \approx \frac{1/2}{2/R_1} = R_1/4$$

Assuming that

$$R + R_1/4 \gg \frac{R_C \cdot \frac{1}{gm}}{R_C + 1/gm}$$

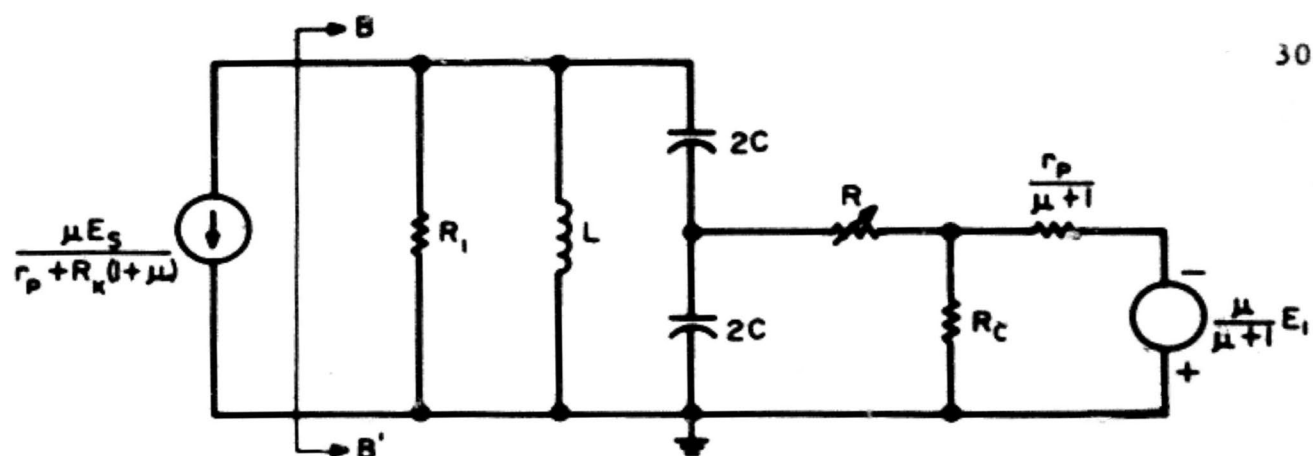


Figure 12. Filter Equivalent Circuit

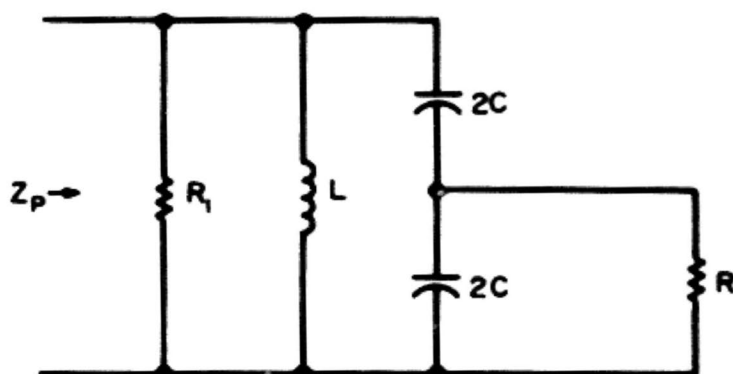


Figure 13. Filter Circuit Without Feedback

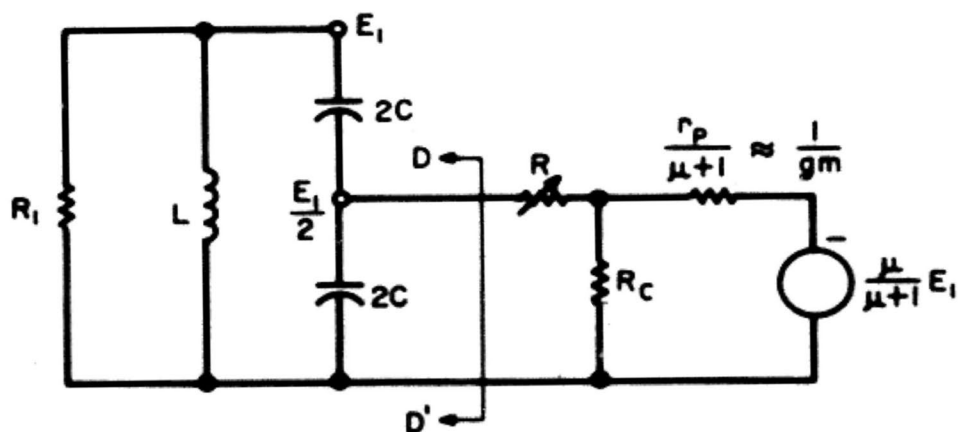


Figure 14. Filter Circuit With Feedback

Then

$$F_{op} = \frac{\mu}{\mu + 1} \frac{R_C}{R_C + 1/gm} \left[\frac{2 R_1/4}{R + R_1/4} \right]$$

$$\frac{\mu}{\mu + 1} \frac{R_C}{R_C + 1/gm} \cong .93$$

$$F_{op} \cong .93 \left[\frac{2R_1}{4R + R_1} \right] = \frac{1.86 R_1}{4R + R_1}$$

$$Z_A = Z_p \left[\frac{1 - F_{sh}}{1 - F_{op}} \right]$$

$$Z_A = \left[\frac{4RR_1}{R_1 + 4R} \right] \left[\frac{1}{1 - 1.86R_1/(4R + R_1)} \right]$$

$$Z_A = \left[\frac{4RR_1}{R_1 + 4R} \right] \left[\frac{R_1 + 4R}{4R + R_1 - 1.86R_1} \right] = R_1 \left[\frac{4R}{4R - .86R_1} \right]$$

The analysis reduced the filter circuit at resonance to a parallel combination of a current source, $\frac{\mu E_s}{r_p + (\mu + 1) R_k}$, and an impedance

$$\frac{4RR_1}{4R - .86R_1}.$$

At resonance the circuit Q_0 is

$$Q_0 = R_p / \omega_0 L$$

For a given Q_0 , $R_p = Q_0 \omega_0 L$

But $R_p = \frac{4RR_1}{4R - .86R_1}$

Therefore $\frac{4RR_1}{4R - .86R_1} = Q_0 \omega_0 L$

$$\frac{4R}{.86R_1} = \frac{Q_0 \omega_0 L}{Q_0 \omega_0 L - R_1}$$

$$R = \frac{.215 R_1 Q_0 \omega_0 L}{Q_0 \omega_0 L - R_1}$$

The above equation gives the necessary value of R to achieve a certain Q_0 , given R_1 and $\omega_0 L$. If the Q_0 is already large enough, that is

$$R_1 = Q_0 \omega_0 L$$

the equation shows that R must be infinite; there is no feedback.

A comparison of analytical and experimental results can be made. The actual spectrum analyzer's one hundred kilocycle channel has an adjusted Q_0 of 3,140. The inductor has an inherent Q_0^1 of 300 and has an inductance of two milli-henry's

$$\omega_0 L = 6.28 \times 10^5 \times 2 \times 10^{-3} = 1,256$$

R_1 is composed of three impedances in parallel, the effective impedance of the input stage, the impedance of the parallel combination of inductor and capacitor and the grid resistors of the cathode followers. This is shown in Figure 15.

The μ of the input stage, the pentode section of a 6678 is about 2,100, its plate resistance is 400 kilohms and its cathode resistor is 30 kilohms.

$$r_p + (1 + \mu) R_k = .4 \text{ Meg} + (2,101) (.03) \text{ Meg} \cong 63 \text{ Meg}$$

The grid resistor is 470 kilohms but it is bypassed to the cathode of the cathode follower which has the effect of multiplying it. As shown in Figure 16, if one volt is applied to the grid the effective voltage across the grid resistor is just .07 volts which makes the current flow through the grid resistor quite small.

$$\text{Input impedance is } E/I = \frac{1 \text{ volt}}{.07 \text{ volt} / 470K}$$

$$E/I = \frac{470K}{.07} \cong 6.7 \text{ Meg}$$

The parallel resistance of the tuned circuit at resonance is

$$\omega_0 L Q_0^1 = 1,256 \times 300 \cong .376 \text{ Meg}$$

$$R_1 = \frac{1}{1/63 + 1/6.7 + 1/.376} \text{ Meg}$$

$$R_1 \cong .356 \text{ Meg} = 3.56 \times 10^5 \text{ ohms}$$

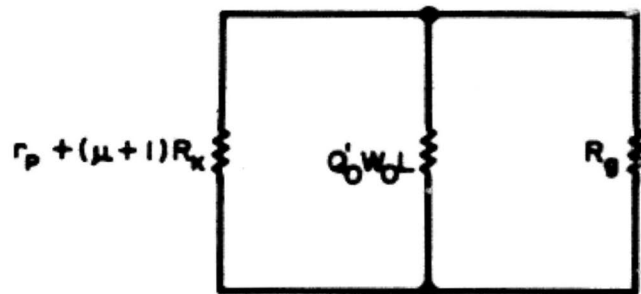


Figure 15. Components of R_1

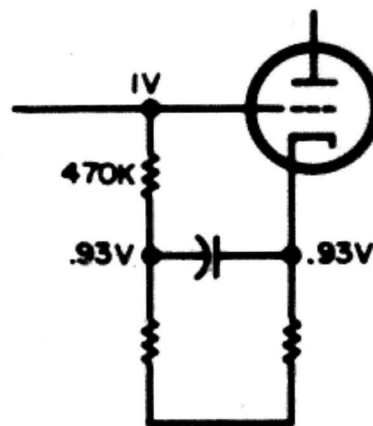


Figure 16. Grid Resistor Multiplication of Cathode Follower

The value of the feedback resistor necessary to obtain the wanted Q_0 is:

$$R = \frac{.215 \times 3.56 \times 10^8 \times 3.94 \times 10^6}{3.94 \times 10^6 - .356 \times 10^6} \approx 8.43 \times 10^6 \text{ ohms}$$

The values of feedback resistors necessary for properly adjusted Q's in the ten kilocycle and four kilocycle channels can also be computed for better comparison to actual values.

For ten kilocycles, $Q_0 = 314$, $L = .1H$, $R_S = 42.2 \text{ ohms}$

$$\omega_0 L Q_0^1 = \frac{\omega_0^2 L^2}{R_S} = \frac{(6.28 \times 10^4)^2 (.1)^2}{42.2} = .935 \times 10^6 \text{ ohms}$$

$$R_1 = \frac{1}{1/63 + 1/6.7 + 1/.935} \text{ Meg}$$

$$R_1 \approx .82 \text{ Meg} = 8.2 \times 10^5 \text{ ohms}$$

$$\omega_0 L Q_0 = (6.28 \times 10^4) (.1) (314) = 1.97 \times 10^6$$

$$R = \frac{.215 \times 8.2 \times 10^5 \times 1.97 \times 10^6}{1.97 \times 10^6 - .82 \times 10^6} \approx 3.02 \times 10^5 \text{ ohms}$$

For four kilocycles, $Q_0 = 126$, $Q_0^1 = 200$, $L = .1H$

$$\omega_0 L Q_0^1 = 6.28 \times 4 \times 10^3 \times 1 \times 2 \times 10^2 = 5.02 \times 10^6$$

At four kilocycles the Q_0^1 was greater than the wanted Q_0 .

Therefore a damping resistor was placed across the inductor to lower the Q_0^1 and allow the circuit Q_0 to be adjusted by the feedback resistor

$$R_1 = \frac{1}{1/63 + 1/6.7 + 1/5.02 + 1/3.9} \approx 1.65 \text{ Meg}$$

$$\omega_0 L Q_0 = (6.28 \times 4 \times 10^3) (1) (126) = 3.16 \times 10^6$$

$$R = \frac{.215 \times 1.65 \times 10^6 \times 3.16 \times 10^6}{3.16 \times 10^6 - 1.65 \times 10^6} \approx 7.43 \times 10^5 \text{ ohms}$$

The feedback resistors in the actual circuits consist of a fixed resistor and a variable resistor in series. They were measured with a Simpson volt-ohmmeter.

Table I shows the correlation between calculated values and actual values of the feedback resistor in three different channels. Considering the various assumptions made in the analysis of the feedback circuit, the results compare quite well.

The Q_0 's of all the spectrum analyzer channels were adjusted so that all of the filter time constants were equal to ten milliseconds. The filter time constant was,

$$T.C. = \frac{2Q_0}{\omega_0} = 10^{-2} \text{ seconds}$$

Therefore
$$\frac{Q_0}{\omega_0} = \frac{10^{-2}}{2} = .005$$

$$Q_0 = .005 \omega_0$$

Thus the theory that the error could be made small by making Q_0 a function of ω_0 was implemented. The maximum error that could occur according to equation (4) could be found

$$\delta \approx \frac{\omega_0}{2Q_0} t = \frac{1}{10^{-2}} \times 10^{-3}$$

$$\delta \approx 0.1 = 10 \text{ per cent}$$

This error was a constant for all frequencies so that the complete spectrum had a 10 per cent smaller amplitude but the relative amplitudes were still correct.

4.2 Auxillary Circuits

An energized high Q filter will ring a long time before all of its energy is dissipated. Thus a relatively long time would have to pass between two signals so that the response to the second was not influenced by the first response. Noise will also energize a high Q filter and affect the response to a waveform.

To avoid these two trouble spots a filter clamp was used. The filter unclamping pulse was applied to a phase splitter and the resulting

TABLE I

COMPARISON OF CALCULATED AND EXPERIMENTAL VALUES OF
FEEDBACK RESISTOR

<u>Frequency (kc)</u>	<u>Calculated R (ohms)</u>	<u>Actual R (ohms)</u>
100	84,300	70,000
10	302,000	450,000
4	743,000	600,000

two signals were applied to the plate of one diode and the cathode of another. Normally these two diodes conducted current between ground and 300 volts through precision resistors to apply 150 volts to one side of the filter. As the other end of the filter was also connected to 150 volts the filter was clamped and could not ring. When the unclamping signals were applied to the diodes they were cut off and the filter was free to respond to a signal. The filter was left unclamped long enough to transfer the information from the filter to the memory and then was clamped which made it ready for another signal.

The filter output amplitude was quite small and so was passed through an amplifier with a gain of ten. The amplifier was of the operational amplifier type to minimize hum at low levels.

A one thousand pico-farad capacitor was the memory. The output of the filter was impressed across the capacitor which charged up to the peak value of the filter response.

A switching circuit was used to couple the filter output to the memory capacitor. Normally the switch was open coupling the filter output to the memory capacitor. No voltage was transferred because the filter was clamped. However, during the interval while the waveform was operating on the filter erroneous results not proportional to the true relative signal energy were generated by the filter. Therefore the switch was opened for the duration of the waveform and then after the filter response was true indication of the waveform's energy the switch was again closed. The memory capacitor could then charge up to the level indicated by the filter.

The difference in lengths of the filter unclamping pulse and the switching pulse was quite important. A peak in the filter response must occur during this time so that the memory could record it. The peak

amplitude of the response of a four kilocycle filter occurred every 250 microseconds. Thus the time difference of 200 microseconds actually present in the prototype machine was not adequate. This was remedied by adding a full wave peak detector. This had the effect of applying the absolute value of the filter response to the memory. A voltage peak thereby occurred every 125 microseconds in the four kilocycle filter.

The memory capacitor was normally clamped to ground potential by the clamp tube due to the small positive voltage on its grid. In fact the plate of the clamp tube would have become negative were it not for the diode across it. The memory capacitor unclamping pulse biased the tube below cutoff which allowed the memory capacitor to charge up. The memory capacitor had 200 microseconds to charge to its peak value, from 1.0 to 1.2 milliseconds, and then retained the information for 700 microseconds.

The last stage was a conventional cathode follower. A variable resistor was in the cathode circuit so that the output could be direct coupled with the base line at ground potential.

A sequential sampler, not part of the spectrum analyzer, was developed that would present the analyzer outputs on an oscilloscope. The sixteen outputs were sampled in sequence and presented in the form of a bar graph. The amplitude frequency spectrum of any input could then be photographed.

4.3 Calibration

A method was needed to adjust the gain of the spectrum analyzer channels. One method was to adjust for a fifty volt output when the input was a five volt, peak to peak, sine wave at the center frequency. This was somewhat difficult because the output would vary depending on the value of the sine wave when the filter was unclamped.

A simpler method which was used, was to develop a test pulse whose spectrum was flat from four kilocycles to beyond one hundred kilocycles. The amplitude frequency spectrum of a one microsecond pulse was down to 98.4 per cent at one hundred kilocycles, but such a narrow pulse could evoke only a small response from the spectrum analyzer. The pulse would be required to produce around a fifty volt output to be a valid test pulse.

When a one microsecond pulse was passed through a one hundred kilocycle low pass filter and amplified to have a eighty volt peak it produced a fifty volt output. Some compensation at seventy-five and one hundred kilocycles made the test pulse have a flat spectrum at the spectrum analyzer frequencies. The spectrum analyzer could be adjusted by setting up one channel by the sine wave method and then adjusting all other channels to produce a flat spectrum.

CHAPTER V

COMPARISON OF ANALYTIC AND EXPERIMENTAL RESULTS

After the spectrum analyzer was constructed some photographs of sampled spectra were taken and compared with calculated spectra to check accuracy. The work in Chapter III derived the lower bound on the accuracy of the spectrum analyzer but the actual results indicated much better accuracies.

5.1 Experimental Results

The spectrum analyzer was calibrated using the test pulse shown in Figure 17 and the method described in Chapter IV. Figure 18 shows the sampled spectrum of the test pulse after the spectrum analyzer was calibrated.

Five different signals whose spectra could be easily calculated were then applied to the input of the spectrum analyzer. Photographs were taken of both the inputs and the outputs. An optical device was then used to increase the size of the photographs and to enable them to be traced on graph paper. The calculated values (see Appendix C) of the spectra for the different inputs were then placed on the graph. The ratio of the difference of the actual and calculated values to the full amplitude value of fifty volts was called the error.

The values of the calculated spectrum corresponding to the sample frequencies are in Table II. Figures 19 through 23 are photographs of the five signals used to check accuracy and their sampled spectra as found by the spectrum analyzer. Figures 24 through 28 are plots of the calculated spectra placed on the graphs of the sampled spectra. Table III compares the experimental to the calculated spectra and also shows the error in each case. Figure 29 is a photograph of simulated waveforms and their sampled spectra.



Figure 17. Test Pulse

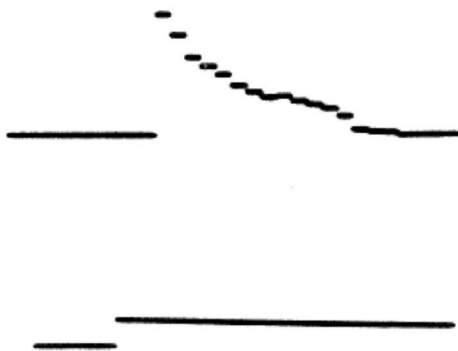
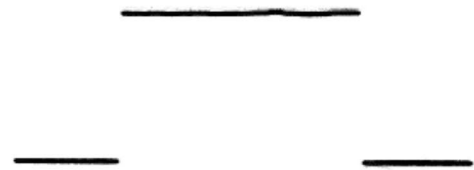


Figure 19. Step Input and Spectrum



Figure 20. 143 μ sec Rectangular Pulse and Spectrum



Figure 21. 100 μ sec Rectangular
Pulse and Spectrum



Figure 22. 83 μ sec Rectangular
Pulse and Spectrum



Figure 23. 40 μ sec Rectangular
Pulse and Spectrum

TABLE II

CALCULATED VALUES OF TEST SPECTRA

Frequency kc	Unit Step	143 μ sec Pulse	100 μ sec Pulse	83 μ sec Pulse	40 μ sec Pulse
0	1.000	1.000	1.000	1.000	1.000
4	.250	.543	.757	.826	.959
5	.200	.365	.637	.738	.936
6	.167	.160	.505	.637	.908
7	.143	0	.368	.528	.876
8	.125	.121	.233	.415	.840
9	.111	.194	.108	.300	.800
10	.100	.217	0	.190	.757
11	.091	.197	.088	.089	.711
12	.083	.144	.156	0	.662
13	.077	.073	.198	.076	.611
14	.071	0	.216	.136	.558
15	.067	.066	.212	.179	.505
25	.040	.087	.127	.039	0
50	.020	.020	0	.038	0
75	.013	.024	.042	.036	0
100	.010	.018	0	.033	0

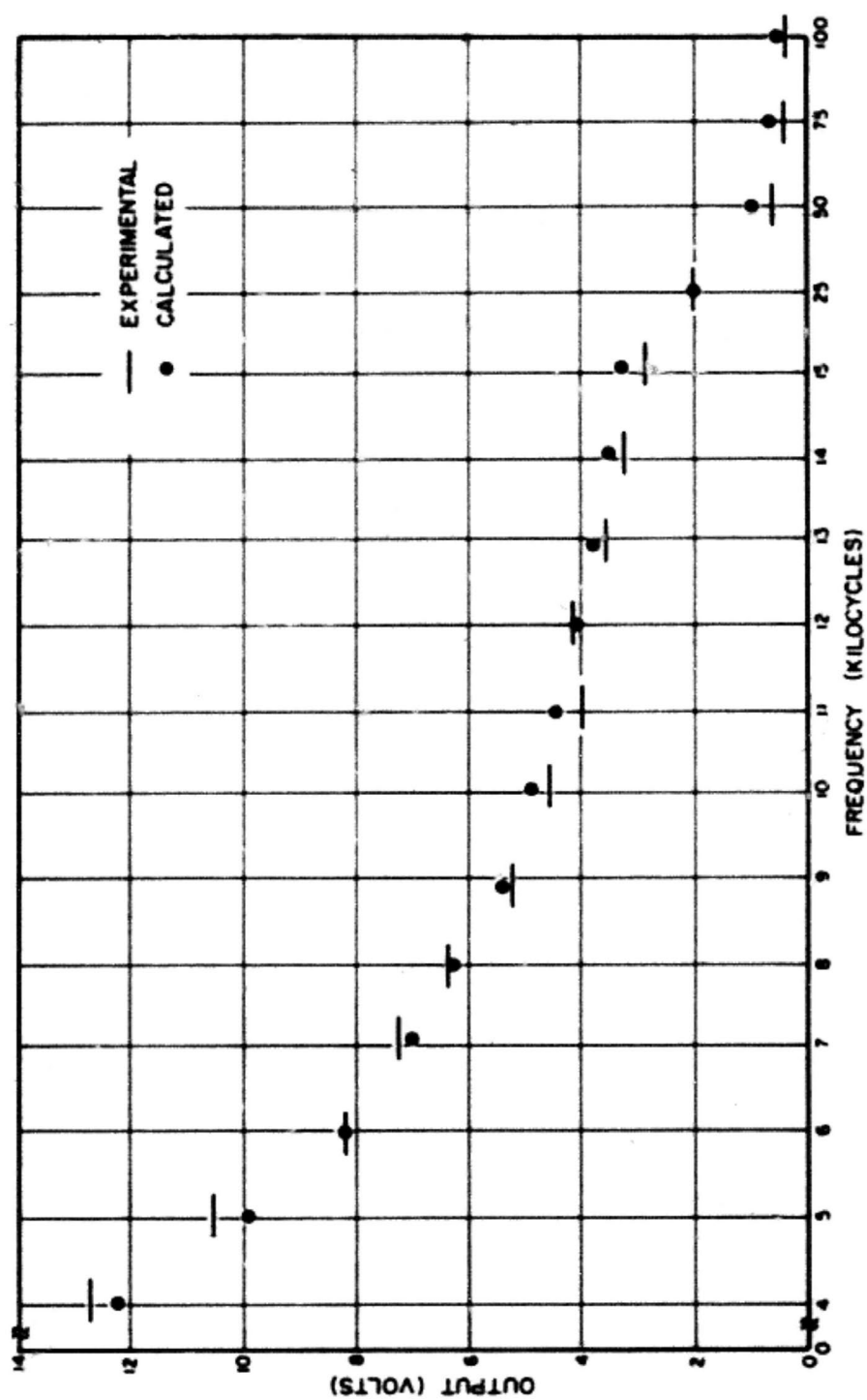


Figure 24. Sampled Spectrum Analyzer Response to a Step Input

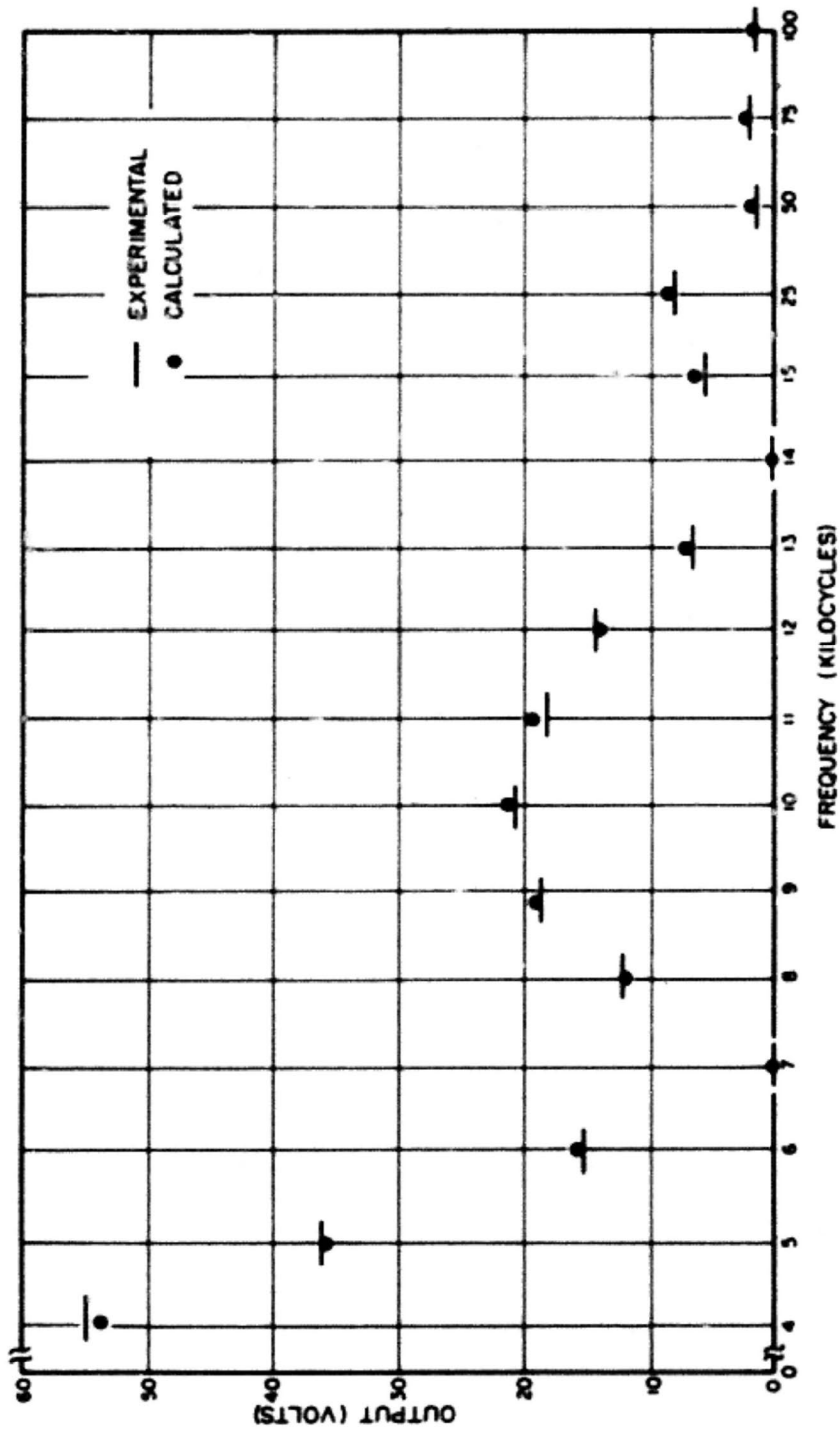


Figure 25. Sampled Spectrum Analyzer Response to a 143 Microsecond Pulse Input

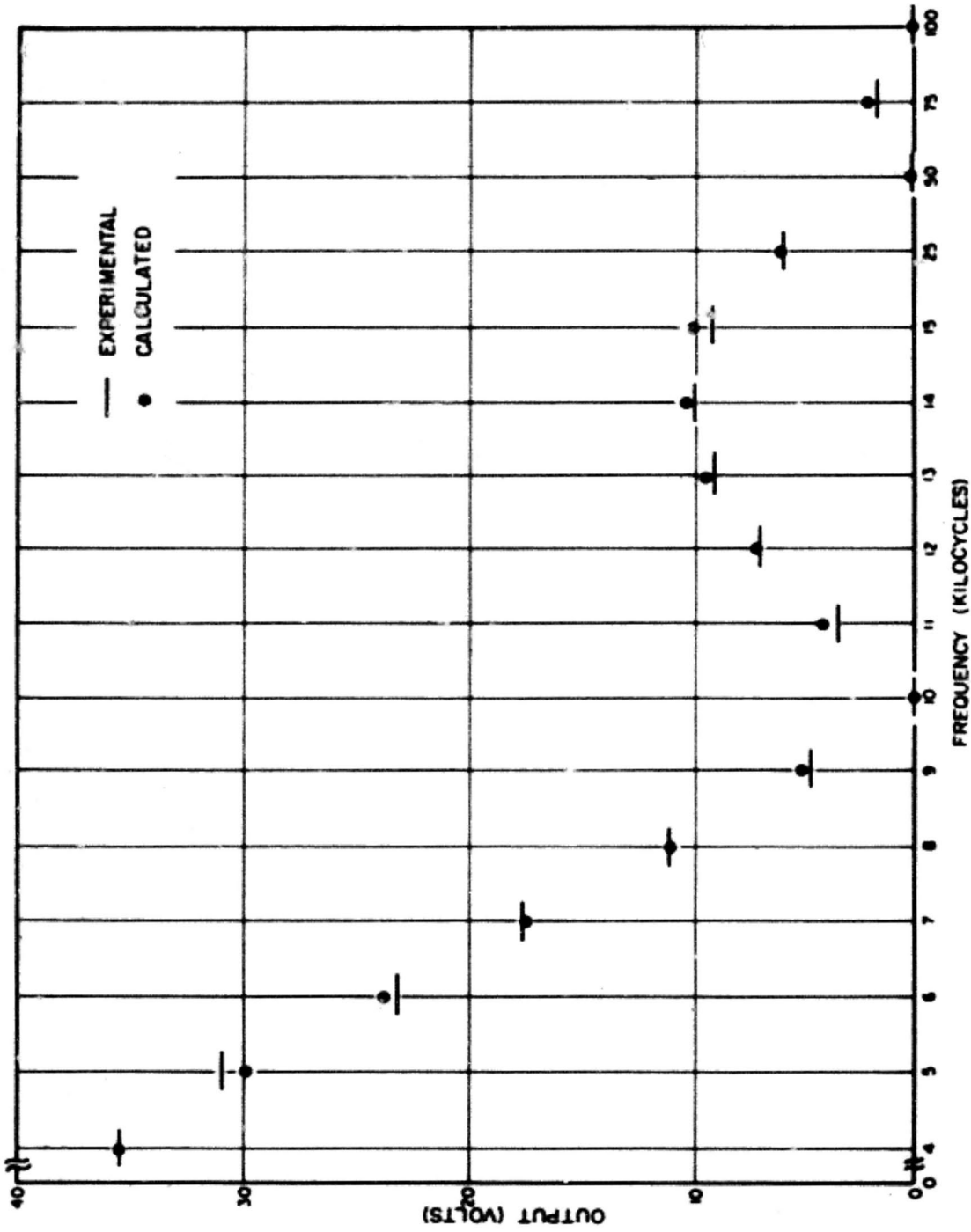


Figure 26. Sampled Spectrum Analyzer Response to a 100 Microsecond Pulse Input

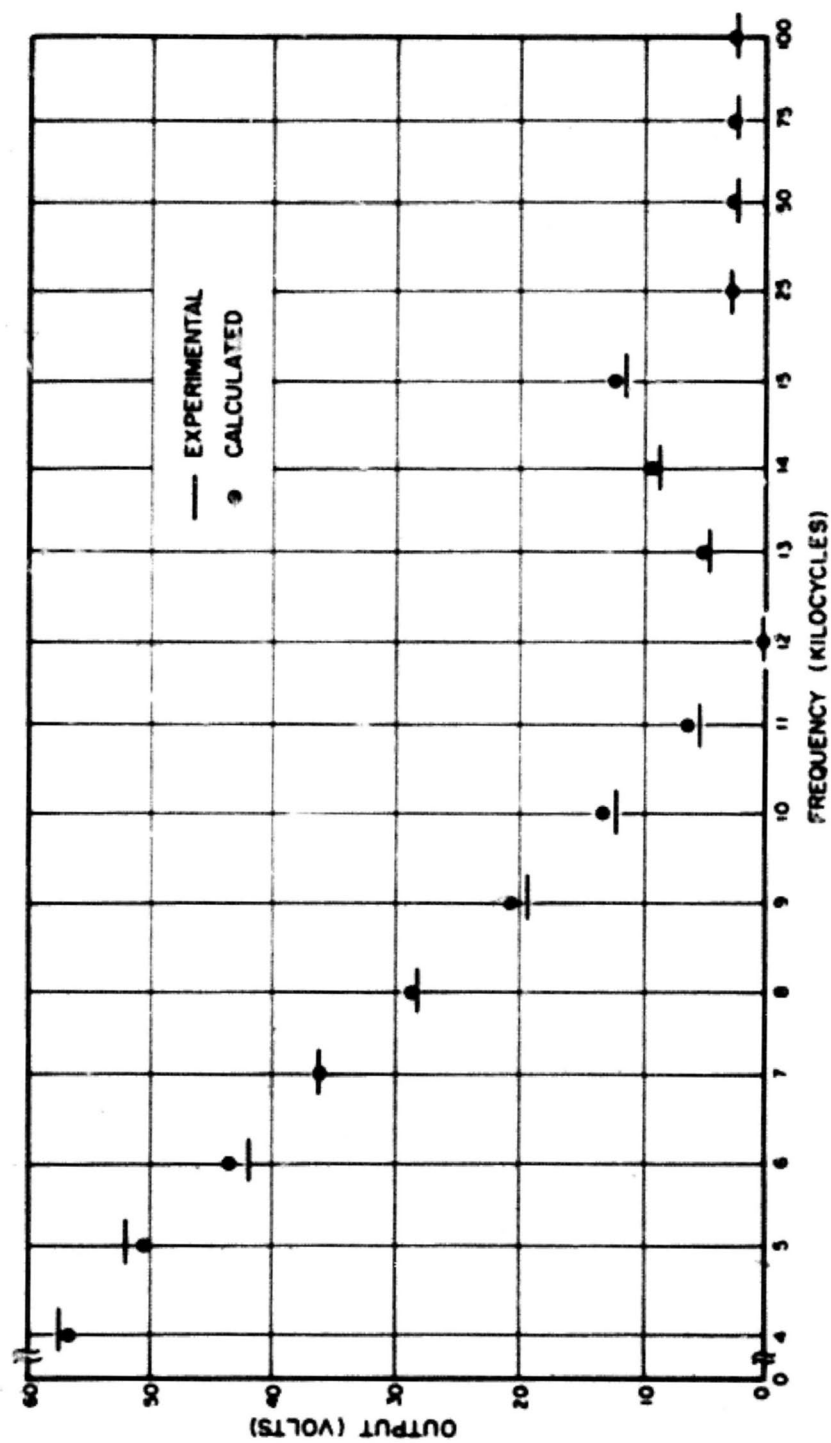


Figure 27. Sampled Spectrum Analyzer Response to a 83 Microsecond Pulse Input

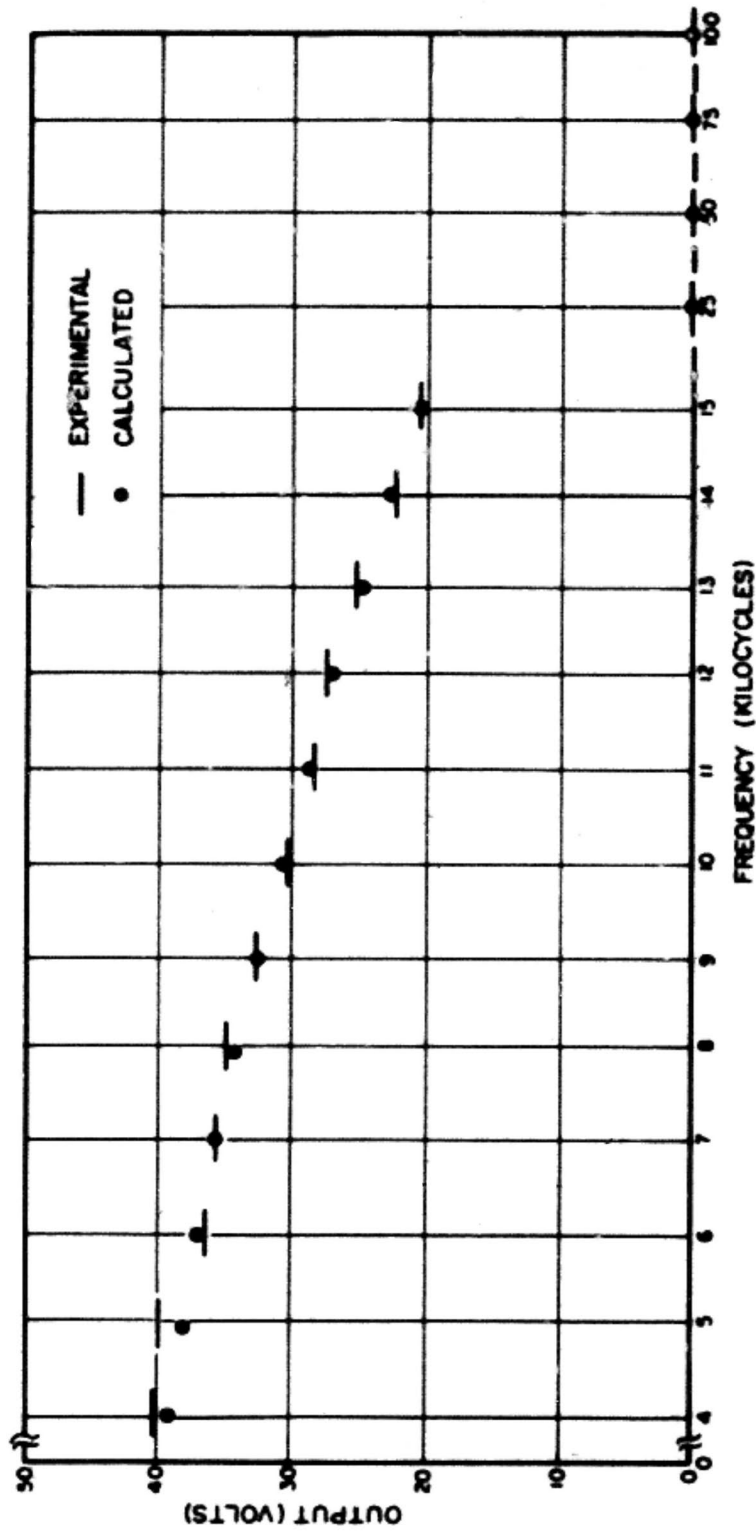


Figure 28. Sampled Spectrum Analyzer Response to a 40 Microsecond Pulse Input

TABLE III

COMPARISON OF CALCULATED
AND EXPERIMENTAL SPECTRA

Freq kc	Step Input			143 μ sec Pulse			100 μ sec Pulse		
	Exp volts	Calc volts	% Er	Exp volts	Calc volts	% Er	Exp volts	Calc volts	% Er
4	12.70	12.22	0.96	54.90	53.32	3.16	35.60	35.60	0
5	10.51	9.77	1.48	36.05	35.90	0.30	31.00	29.98	2.04
6	8.16	8.17	0	15.31	15.73	0.84	23.15	23.75	1.20
7	7.24	7.00	0.48	0	0	--	17.52	17.31	0.42
8	6.35	6.12	0.46	12.20	11.87	0.66	10.98	10.96	0.04
9	5.20	5.42	0.44	18.65	19.08	0.86	4.65	5.07	0.84
10	4.55	4.88	0.66	20.68	21.33	1.30	0	0	--
11	3.86	4.46	1.20	18.12	19.34	2.40	3.45	4.08	1.26
12	4.10	4.06	0.10	14.32	14.18	0.28	6.88	7.32	0.88
13	3.54	3.76	0.44	6.54	7.16	1.24	8.92	9.25	0.66
14	3.21	3.47	0.52	0	0	--	9.72	10.15	0.86
15	2.82	3.28	0.92	5.49	6.48	2.02	8.97	9.97	2.00
25	1.99	1.95	0.10	7.95	8.57	1.24	5.80	5.96	0.32

TABLE III (Cont.)

COMPARISON OF CALCULATED
AND EXPERIMENTAL SPECTRA

Freq kc	Step Input		143 μ sec Pulse			100 μ sec Pulse		
	Exp volts	Calc volts	% Er	Exp volts	Calc volts	% Er	Exp volts	Calc volts
50	0.57	0.97	0.80	1.46	1.99	1.06	0	0
75	0.36	0.63	0.54	2.09	2.35	0.48	1.60	1.99
100	0.32	0.49	0.34	1.46	1.78	0.64	0	0

TABLE III (Cont.)
COMPARISON OF CALCULATED
AND EXPERIMENTAL SPECTRA

Freq kc	83 μ sec Pulse			40 μ sec Pulse		
	Exp volts	Calc volts	% Er	Exp volts	Calc volts	% Er
4	57.30	56.40	1.80	40.30	38.92	2.76
5	52.00	50.40	3.20	39.90	38.00	3.80
6	41.90	43.50	3.20	36.34	36.82	0.96
7	36.16	36.05	0.22	35.40	35.56	0.32
8	28.30	29.34	2.08	34.80	34.10	1.40
9	19.30	20.48	2.36	32.58	32.46	0.24
10	12.08	12.98	1.80	30.32	30.77	0.90
11	5.32	6.01	1.38	28.37	28.85	0.96
12	0	0	--	27.42	26.84	1.16
13	4.53	5.18	1.30	25.21	24.80	0.84
14	8.57	9.28	1.42	22.80	22.64	0.32
15	11.28	12.23	1.90	20.40	20.50	0.20
25	2.66	2.68	0.04	0	0	--
50	2.13	2.59	0.92	0	0	--
75	2.13	2.48	0.70	0	0	--
100	2.13	2.25	0.24	0	0	--



Figure 29. Three Simulated Waveforms and their Spectra

CHAPTER VI

CONCLUSION

An error analysis was performed on the transient signal frequency spectrum analyzer in question. The equations of its operation were written and the possible errors due to circuit losses and different types of inputs were calculated.

The maximum error occurred when the input was a step function. A 10 percent error was possible when a channel's actual output was compared to a lossless circuit's theoretical output. However, if each channel had the same 10 percent error, the sampled spectrum relative amplitude should be correct.

Different types of pulses whose spectra were easily calculated were used as inputs to obtain test spectra. Photographs of these spectra were taken and the errors checked by comparing them to calculated spectra. The errors in the spectra measured were never greater than 3.8 percent. Therefore the conclusion of this error analysis is that the transient signal frequency spectrum analyzers sampled spectra have less than 5 percent error.

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BIBLIOGRAPHY

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**AN ERROR ANALYSIS OF A PROTOTYPE FREQUENCY SPECTRUM
ANALYZER FOR SIGNALS OF SHORT DURATION**

**An Abstract of a Thesis
Presented to
The Faculty of the Graduate College
University of Denver**

**In Partial Fulfillment
of the Requirements for the Degree
Master of Science**

**by
Edwin Joseph Tajchman
August 1961**

ABSTRACT

A transient signal frequency spectrum analyzer using a technique different than that ordinarily used was analyzed. By use of the superposition integral equations were written describing the operations performed by the analyzer and compared to the Fourier integral.

Possible errors due to circuit losses and different types of inputs were calculated. The lower bound on the relative accuracy of a single channel was established. The accuracy of the total sampled spectrum was much higher as a method was developed to keep the relative error nearly constant in all channels.

The sampled frequency spectra of certain pulses were compared to the pulses calculated spectra. The error in the sampled spectrum was always less than 5 percent.

APPENDIX

APPENDIX A

SAMPLING THEOREM IN FREQUENCY DOMAIN

The frequency spectrum of a transient waveform such as the one in Figure 30 can be found by the Fourier integral theorem.¹⁰

$$F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt$$

Or

$$F(\omega) = \int_{T_1}^{T_2} f(t) e^{-j\omega t} dt$$

A Fourier series expansion of $f(t)$ in the interval T_1 to T_2 can be written as;¹¹

$$f(t) = \sum_{n=-\infty}^{n=+\infty} C_n e^{j\left(\frac{2\pi n}{T_2-T_1}\right)t}$$

Where

$$C_n = \frac{1}{T_2-T_1} \int_{T_1}^{T_2} f(t) e^{-j\left(\frac{2\pi n}{T_2-T_1}\right)t} dt$$

$$C_n = \frac{F\left(\frac{2\pi n}{T_2-T_1}\right)}{T_2-T_1}$$

Therefore

$$F(\omega) = \int_{T_1}^{T_2} \left[\sum_{n=-\infty}^{n=+\infty} \frac{F\left(\frac{2\pi n}{T_2-T_1}\right)}{T_2-T_1} e^{j\left(\frac{2\pi n}{T_2-T_1}\right)t} \right] e^{-j\omega t} dt$$

$$F(\omega) = \sum_{n=-\infty}^{n=+\infty} \frac{F\left(\frac{2\pi n}{T_2-T_1}\right)}{T_2-T_1} \int_{T_1}^{T_2} e^{-j\left(\omega - \frac{2\pi n}{T_2-T_1}\right)t} dt$$

This can be simplified if $T_1 = -T/2$ and $T_2 = T/2$ so that $T_2-T_1 = T$

Then

$$F(\omega) = \sum_{n=-\infty}^{n=+\infty} \frac{F\left(\frac{2\pi n}{T}\right)}{T} \int_{-T/2}^{T/2} e^{-j\left(\omega - \frac{2\pi n}{T}\right)t} dt$$

¹⁰S. Goldman, Frequency Analysis, Modulation and Noise (New York: McGraw-Hill Book Company, Inc., 1948) p. 61.

¹¹Ibid. p23.

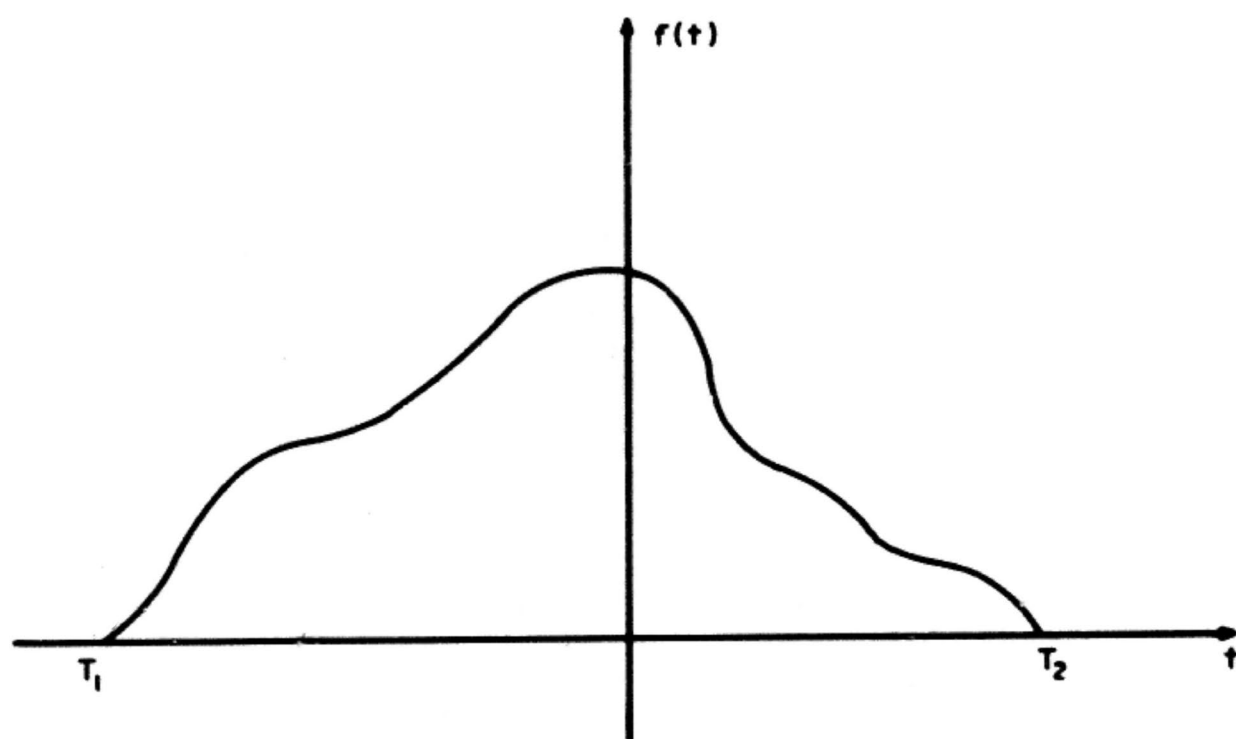


Figure 30. A Transient Waveform

Expanding the exponential,

$$F(\omega) = \sum_{n=-\infty}^{n=+\infty} \frac{F(\frac{2\pi n}{T})}{T} \int_{-T/2}^{T/2} \left[\cos(\omega - \frac{2\pi n}{T})t - j \sin(\omega - \frac{2\pi n}{T})t \right] dt$$

Since the absolute values of the limits are equal, the sine term equals zero and the integral can be written as,

$$F(\omega) = 2 \sum_{n=-\infty}^{n=+\infty} \frac{F(\frac{2\pi n}{T})}{T} \int_0^{T/2} \cos(\omega - \frac{2\pi n}{T})t \, dt$$

$$F(\omega) = 2 \sum_{n=-\infty}^{n=+\infty} \frac{F(\frac{2\pi n}{T})}{T} \frac{\sin(\omega - \frac{2\pi n}{T})t}{(\omega - \frac{2\pi n}{T})} \Big|_0^{T/2}$$

$$F(\omega) = \sum_{n=-\infty}^{n=+\infty} F(\frac{2\pi n}{T}) \frac{\sin T/2 (\omega - \frac{2\pi n}{T})}{T/2 (\omega - \frac{2\pi n}{T})}$$

$F(\frac{2\pi n}{T})$ is the value of the Fourier integral of $f(t)$ at the frequency $\omega = \frac{2\pi n}{T}$. This corresponds to an individual spectrum analyzer sample.

The equation for $F(\omega)$ states that the frequency spectrum of a transient wave contained in time interval T can be found by an infinite sum of samples spaced $1/T$ cycles apart. Thus the frequency spectrum of a waveform contained in an interval of one millisecond can be found by taking samples every kilocycle from ω equals minus infinity to ω equals positive infinity.

APPENDIX B

EVALUATION OF OUTPUT INTEGRAL WITH GATED SINEWAVE INPUT

$$I(\tau) = \sin \omega \tau$$

$$E_o(t) = \frac{e^{-at}}{c} \left\{ \left(I_1 + \frac{a}{\omega} I_2 \right) \cos \omega t + \left(I_2 - \frac{a}{\omega} I_1 \right) \sin \omega t \right\}$$

Where

$$I_1 = \int_0^T I(\tau) \cos \omega \tau e^{a\tau} d\tau \text{ and}$$

$$I_2 = \int_0^T I(\tau) \sin \omega \tau e^{a\tau} d\tau$$

To evaluate I_1 :

$$I_1 = \int_0^T \sin \omega \tau \cos \omega \tau e^{a\tau} d\tau$$

$$I_1 = \frac{1}{2} \int_0^T \sin 2\omega \tau e^{a\tau} d\tau$$

$$I_1 = \frac{e^{a\tau}}{2} \frac{a \sin 2\omega \tau - 2\omega \cos 2\omega \tau}{a^2 + 4\omega^2} \Big|_0^T$$

$$I_1 = \frac{e^{aT}}{2} \frac{(a \sin 2\omega T - 2\omega \cos 2\omega T)}{a^2 + 4\omega^2} - \frac{e^0}{2} \frac{(a \sin 0 - 2\omega \cos 0)}{a^2 + 4\omega^2}$$

$$I_1 = \frac{e^{aT}}{2} \frac{(a \sin 2\omega T - 2\omega \cos 2\omega T)}{a^2 + 4\omega^2} + \frac{\omega}{a^2 + 4\omega^2}$$

Since

$$T = \frac{N2\pi}{\omega_0}$$

$$\sin 2\omega T = \sin \frac{2\omega N2\pi}{\omega_0} \approx \sin 4N\pi = 0$$

$$\cos 2\omega T = \cos \frac{2\omega N2\pi}{\omega_0} \approx \cos 4N\pi = 1$$

Therefore,

$$I_1 = \frac{-e^{aT} 2\omega}{2(a^2 + 4\omega^2)} + \frac{\omega}{a^2 + 4\omega^2}$$

$$I_1 = \frac{-\omega}{a^2 + 4\omega^2} (e^{aT} - 1)$$

To evaluate I_2 ,

$$I_2 = \int_0^T \sin^2 \omega \tau e^{a\tau} d\tau$$

$$I_2 = \frac{1}{2} \int_0^T e^{a\tau} d\tau - \frac{1}{2} \int_0^T \cos \omega \tau e^{a\tau} d\tau$$

$$I_2 = \frac{1}{2a} e^{a\tau} \Big|_0^T - e^{a\tau} \frac{(a \cos 2\omega \tau + 2\omega \sin 2\omega \tau)}{a^2 + 4\omega^2} \Big|_0^T$$

$$I_2 = \frac{e^{aT} - e^0}{2a} - e^{aT} \frac{(a \cos 2\omega T + 2\omega \sin 2\omega T)}{2(a^2 + 4\omega^2)} + e^0 \frac{(a \cos 0 + 2\omega \sin 0)}{2(a^2 + 4\omega^2)}$$

$$I_2 = \frac{e^{aT} - 1}{2a} - e^{aT} \frac{(a \cos 2\omega T + 2\omega \sin 2\omega T)}{2(a^2 + 4\omega^2)} + \frac{a}{2(a^2 + 4\omega^2)}$$

$$I_2 = \frac{e^{aT} - 1}{2a} - \frac{a(e^{aT} - 1)}{2(a^2 + 4\omega^2)}$$

$$I_2 = \frac{e^{aT} - 1}{2} \left[\frac{1}{a} - \frac{a}{a^2 + 4\omega^2} \right]$$

$$I_2 = (e^{aT} - 1) \left[\frac{2\omega^2}{a(a^2 + 4\omega^2)} \right]$$

Now

$$a(\omega) = I_1 + \frac{a}{\omega} I_2$$

$$a(\omega) = (e^{aT} - 1) \frac{-\omega}{a^2 + 4\omega^2} + \frac{a}{\omega} (e^{aT} - 1) \frac{2\omega^2}{a(a^2 + 4\omega^2)}$$

$$a(\omega) = \frac{(e^{aT} - 1) \omega}{a^2 + 4\omega^2}$$

$$b(\omega) = I^2 - \frac{a}{\omega} I_1$$

$$b(\omega) = (e^{aT} - 1) \frac{2\omega^2}{a^2 + 4\omega^2} + \frac{a}{\omega} \frac{(e^{aT} - 1) \omega}{a^2 + 4\omega^2}$$

$$b(\omega) = \frac{(e^{aT} - 1)}{a^2 + 4\omega^2} \left[\frac{2a^2}{a} + a \right] = (e^{aT} - 1) \frac{2\omega^2 + a^2}{a(a^2 + 4\omega^2)}$$

Therefore,
$$E_0(t) = \frac{e^{-at}}{C} \left[\frac{(e^{aT} - 1) \omega}{a^2 + 4\omega^2} \cos \omega t + \frac{(e^{aT} - 1)(2\omega^2 + a^2)}{a(a^2 + 4\omega^2)} \sin \omega t \right]$$

$$E_0(t) = \frac{e^{-at}}{C} \frac{(e^{aT} - 1)}{a^2 + 4\omega^2} \left\{ \omega \cos \omega t + \frac{(2\omega^2 + a^2)}{a} \sin \omega t \right\}$$

$$E_0(t) = \frac{e^{-at} (e^{aT} - 1)}{C(a^2 + 4\omega^2)} \sqrt{\omega^2 + \frac{(2\omega^2 + a^2)^2}{a^2}} \sin(\omega t + \phi), \phi = \tan^{-1} \frac{\omega a}{2\omega^2 + a^2}$$

$$\omega^2 + \frac{(2\omega^2 + a^2)^2}{a^2} = \frac{\omega^2 a^2 + 4\omega^4 + 4\omega^2 a^2 + a^4}{a^2}$$

But

$$a = \frac{\omega_0}{2Q_0}$$

So

$$\omega^2 + \frac{(2\omega^2 + a^2)^2}{a^2} = \frac{\frac{\omega^2 \omega_0^2}{4Q_0^2} + 4\omega^4 + \frac{4\omega^2 \omega_0^2}{4Q_0^2} + \frac{\omega_0^4}{16Q_0^4}}{a^2}$$

$$\omega \approx \omega_0$$

$$\omega^2 + \frac{(2\omega^2 + a^2)^2}{a^2} = \frac{\omega^4}{a^2} \left(\frac{1}{4Q_0^2} + 4 + \frac{1}{Q_0^2} + \frac{1}{16Q_0^2} \right)$$

$$\approx \frac{4\omega^4}{a^2}$$

Since $Q_0 \geq 10$

Therefore $E_0(t) = \frac{e^{-at} (e^{aT} - 1)}{C(a^2 + 4\omega^2)} \sqrt{\frac{4\omega^4}{a^2}} \sin(\omega t + \phi)$

$$E_0(t) = \frac{e^{-at} (e^{aT} - 1)}{Ca} \frac{2\omega^2}{a^2 + 4\omega^2} \sin(\omega t + \phi)$$

$$\frac{2\omega^2}{a^2 + 4\omega^2} = \frac{2\omega^2}{\omega_0^2/4Q_0^2 + 4\omega^2} \approx \frac{2}{1/4Q_0^2 + 4} \approx \frac{1}{2}$$

$$\phi = \frac{\omega a}{2\omega^2 + a^2} = \frac{\omega \omega_0 / 2Q_0}{2\omega^2 + \omega_0^2 / 4Q_0^2}$$

$$\phi = \frac{\omega^2 / 2Q_0}{2\omega^2 + \omega_0^2 / 4Q_0^2} \approx \frac{1/2Q_0}{2 + 1/4Q_0^2} \approx \frac{1}{4Q_0}$$

$$\phi \leq \frac{1}{40} \text{ as } Q_0 \geq 10$$

$$\phi \leq .025 \text{ radians} = 1.4 \text{ degrees}$$

Therefore ϕ can be ignored.

$$E_0(t) \approx \frac{e^{-at} (e^{aT} - 1)}{2aC} \sin \omega t$$

APPENDIX C

METHOD OF CALCULATING FREQUENCY SPECTRA OF TEST PULSES

The frequency spectra of various types of waveforms can be calculated for comparison to spectra obtained by use of the spectrum analyzer. The frequency spectrum of a rectangular pulse is given by;¹²

$$S(\omega) = \frac{2}{\omega} \sin\left(\frac{\omega\tau}{2}\right)$$

Calculating the amplitude of the four kilocycle component of the frequency spectrum of a one hundred microsecond rectangular pulse gives;

$$S(4KC) = \frac{2}{4 \times 10^3 \times 6.28} \sin\left(\frac{4 \times 10^3 \times 6.28 \times 10^{-4}}{2}\right)$$

$$S(4KC) = \frac{10^{-3}}{12.56} \sin(1.256)$$

$$S(4KC) = .757 \times 10^{-4}$$

Now¹³

$$S(0) = T$$

When

$$T = 10^{-4}, S(0) = 10^{-4}$$

Only the relative frequency spectrum is needed for comparison to the spectrum analyzer output. Therefore the spectrum can be normalized by multiplying each term by 10^4 . Then,

$$S(0) = 1$$

$$S(4KC) = .757 \text{ and so on.}$$

¹²S Goldman, Frequency Analysis, Modulation and Noise (New York: McGraw-Hill Book Company, Inc., 1948) p. 61.

¹³Ibid., p. 62.

The frequency spectrum of a unit step function can be calculated from¹⁴

$$S(\omega) = \frac{1}{\omega}$$

A normalized spectrum then is

$$S(4KC) = .250$$

$$S(5KC) = .200 \text{ and so on.}$$

¹⁴Ibid., p. 125.

